EE 360 Discussion: Solution to Quiz 6

1. Find the Zero-Input Response (ZIR) of the system described by:

$$y[n] - 2r\cos(w_0)y[n-1] + r^2 y[n-2] = r\sin(w_0)\,x[n-1]$$

Where r and w_0 are real values, $|r|<1$, and the following states set for the output

$$y[n=0]=0,\quad y[n=-1] = -r^{-1}\sin(w_0),\ \text{and}\quad y[n=-2] = -r^{-2}\sin(2w_0)$$

■ To begin, note that the inputs of a ZIR are all set to zero, or:

$$\underline{x[n] = 0,\quad n \in \mathbb{R}}$$

We can rewrite the expression from the beginning as:

$$y[n] - 2r\cos(w_0)y[n-1] + r^2 y[n-2] = 0 \quad (1)$$

From Complex analysis, we recall that the expression above will have a solution, which can be hypothesized by:

$$y[n] = \sum_{i=1}^{N} c_i \lambda_i^n \quad (2) \rightarrow \lambda: \text{characteristic variable}$$

One may recognize that the system takes the form of an auxillary eqn. By doing a practical substitution:

$$y[n-N] \Rightarrow \lambda^{-N} \quad (3)$$

The system in (1) is expressed as:

$$1 - 2r\cos(w_0)\lambda^{-1} + r^2\lambda^{-2} = 0 \quad (4) \text{ (auxillary equation)}$$

The auxillary eqn can also be expressed in an alternative form by multiplying the characteristic variable into ④ to create a quadratic eqn:

$$\lambda^2 - 2r\cos(\omega_0)\lambda + r^2 = 0 \quad ⑤$$

From Complex analysis, recall the Exponential form of Cosine:

$$\cos(\phi) = \frac{1}{2}[e^{+j\phi} + e^{-j\phi}] \quad ⑥$$

Substituting ⑥ into ⑤:

$$\lambda^2 - \cancel{2}r\left[\frac{1}{\cancel{2}}\{e^{+j\omega_0} + e^{-j\omega_0}\}\right]\lambda + r^2 = 0$$

From Linear Algebra, we can use decomposition to factor out the expression above:

$$\lambda^2 - r\lambda e^{+j\omega_0} - re^{-j\omega_0}\lambda + r^2 = 0$$

"λ" is a common factor

"$-re^{-j\omega_0}$" is a common factor

$$\lambda(\lambda - re^{+j\omega_0}) - re^{-j\omega_0}(\lambda - re^{+j\omega_0}) = 0$$

Common terms

so

$$(\lambda - re^{-j\omega_0})(\lambda - re^{+j\omega_0}) = 0 \Rightarrow \lambda = re^{\pm j\omega_0} \quad ⑦$$

a general solution can now be expressed by plugging ⑦ into ②

$$Y_{ZIR}[n] = C_1 \lambda_1^n + C_2 \lambda_2^n$$
$$= C_1 (r e^{+j\omega_0})^n + C_2 (r e^{-j\omega_0})^n \quad ⑧$$

we begin to solve for C_1 and C_2:
using the state:

$$y[n=0] = 0 = C_1 (r e^{j\omega_0})^0 + C_2 (r e^{-j\omega_0})^0$$
$$\therefore \quad C_1 = -C_2 \quad ⑨ \rightarrow C_2 = -C_1$$

$$y[n=-1] = -r^{-1} \sin(\omega_0) = C_1 (r e^{j\omega})^{-1} + C_2 (r e^{-j\omega})^{-1}$$
$$-r^{-1} \sin(\omega_0) = C_1 r^{-1} e^{-j\omega_0} + C_2 r^{-1} e^{+j\omega_0}$$

substituting ⑧ into the above:

$$+ r^{-1} \sin(\omega_0) = -C_1 r^{-1} e^{-j\omega_0} + C_1 r^{-1} e^{+j\omega_0}$$
$$= C_1 r^{-1} [r^{-1} e^{+j\omega} - r^{-1} e^{-j\omega_0}]$$

Recall the Exponential form of sine: $\sin(\phi) = \frac{1}{2j}[e^{+j\omega} - e^{-j\omega}]$

The expression is then:

$$\cancel{r^{-1}} \sin(\omega_0) = C_1 \cancel{r^{-1}} [2j \sin(\omega_0)]$$

$$\therefore \quad C_1 = \frac{1}{2j}$$

Recall ⑨:

$$C_2 = \frac{-1}{2j}$$

And now plugging the coefficients into the General Solution: (8)

$$y_{ZIR}[n] = \left(\frac{1}{2j}\right)\left(re^{+j\omega_0}\right)^n + \left(-\frac{1}{2j}\right)\left(re^{-j\omega_0}\right)^n$$

$$= \frac{1}{2j} r^n \left[e^{+j\omega_0 n} - e^{-j\omega_0 n}\right]$$

sine Expression!

Finally

$$\boxed{y_{ZIR}[n] = r^n \sin\left(\frac{\omega_0 n}{\cancel{\omega_0}}\right) u(n)}$$

unitstep function