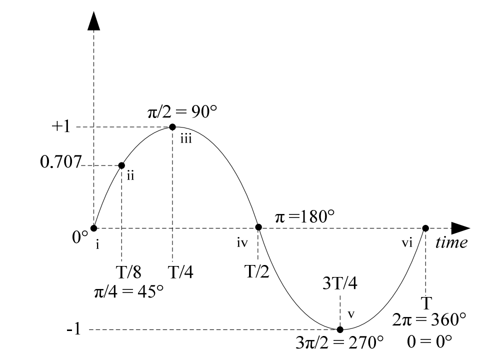
Lincoln Bollschweiler

1.1 Suggest an alternate physical example to the swinging pendulum of Fig. 1.1 to describe sinusoidal motion.

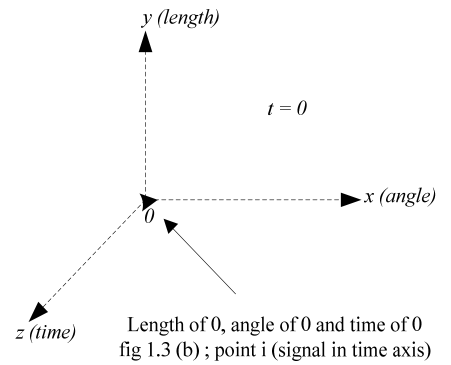
One example could be a weight attached to the end of a spring. When it is released it is at the top of a sinusoidal curve and the force of gravity is much larger than the force of the spring. Thus, it begins to accelerate downwards. It reaches its maximum velocity as the sinusoid crosses zero. This is where the force of gravity and the force of the spring are equal. Now it begins to decelerate. At the bottom of the curve the spring has stopped expanding, the force of the spring is much larger than the force of gravity, and the weight stops falling. It then begins to accelerate upwards, crossing zero at its maximum velocity upwards, and finally comes to rest at the top of the sinusoidal curve, completing one full period. The velocity of the spring, when plotted against time, will take on the shape of a sinusoid.

Q 1.2 Add the z-axis (time) to the representations of the sine wave seen in Figs. 1.3b-f as discussed.

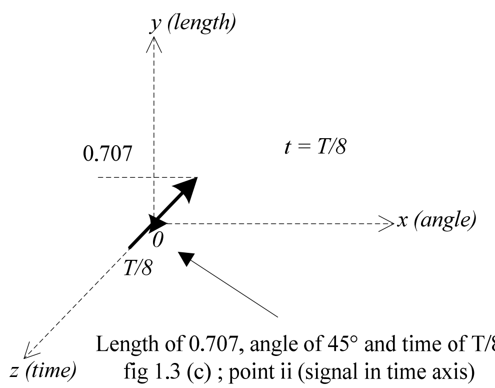
Sol. Figures below shows the addition of z axis (time) to the figures 1.3b-f as discussed in the chapter. The amplitude of "sine" function is plotted based on the five data points at different time interval for time period of signal equal to T.



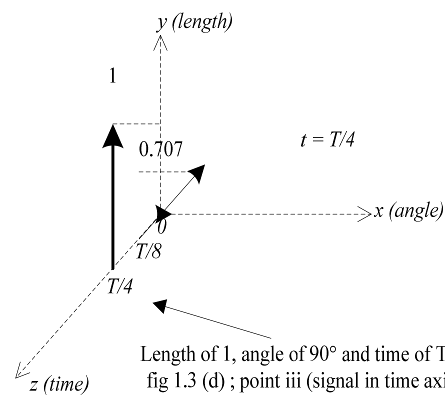
From fig 1.3 (a) ; sine wave 6 points



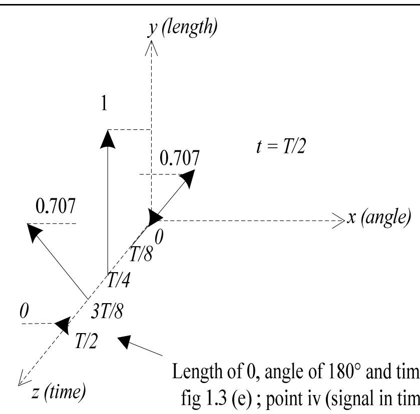
Length of 0, angle of 0 and time of 0
fig 1.3 (b) ; point i (signal in time axis)



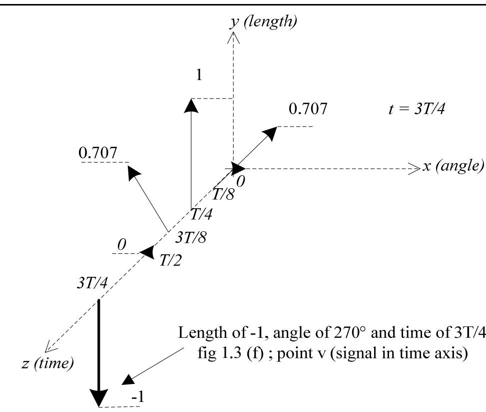
Length of 0.707, angle of 45° and time of T/8
fig 1.3 (c) ; point ii (signal in time axis)



Length of 1, angle of 90° and time of T/4
fig 1.3 (d) ; point iii (signal in time axis)

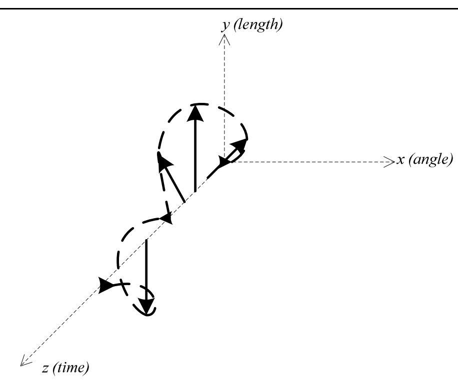


Length of 0, angle of 180° and time of T/2
fig 1.3 (e) ; point iv (signal in time axis)



Length of -1, angle of 270° and time of 3T/4
fig 1.3 (f) ; point v (signal in time axis)

The complete signal will look something as show in adjacent figure, a "cork screw" waveform.



Tyler Hansen

<u>Problem 1.3</u>

Suppose an IQ signal is generated using an in-phase component having an amplitude of 0.5V and a quadrature component having an amplitude of 1V. Sketch the resulting waveform, the IQ signal, in the time domain.

I will arbitrarily assign a cosine as my In-phase carrier, and a sine as my Quadrature carrier. Just as easily I could have assigned sine as my In-phase and cosine as my Quadrature… the only requirement is that the 2 signals maintain a phase-shift of 90 degrees.

As shown in Figure 1a) I can represent my In-phase signal in the complex plane with a vector along the Real axis with a length of 0.5, and my quadrature signal as a vector along the Imaginary axis with length 1.0. Figure 1b) shows how the In-phase and Quadrature signals combine to form the IQ signal which will be transmitted. This is represented as a sinusoid with magnitude according to Equation 1.3.1, and phase according to Equation 1.3.2.

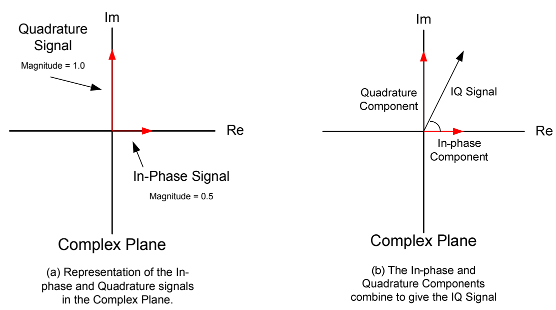


Figure 1.3.1: Complex Plane representation of the phase relationship of IQ signals, and the method for combining them into a single IQ signal in preparation for transmission.

$$Magnitude = \sqrt{A_{Inphase}{}^2 + A_{Quadrature}{}^2} = \sqrt{0.5^2 + 1^2} = \sqrt{1.25} = 1.118 \qquad 1.3.1$$

$$Phase = \tan^{-1}\left(\frac{A_Q}{A_I}\right) = \tan^{-1}\frac{1}{0.5} = 63.4° \qquad 1.3.2$$

Now it is a simple matter to plot the 3 signals in the time domain and look at their relationships. The In-phase signal is a cosine with an amplitude of 0.5. The Quadrature signal is a sin with an amplitude of 1. And the IQ signal is the addition of the 2 signals at every value t. This is done in Figure 1.3.2 below. Note that the frequency is represented arbitrarily. The frequency doesn't matter in our plot, but it is important to note that all 3 signals have the same frequency.

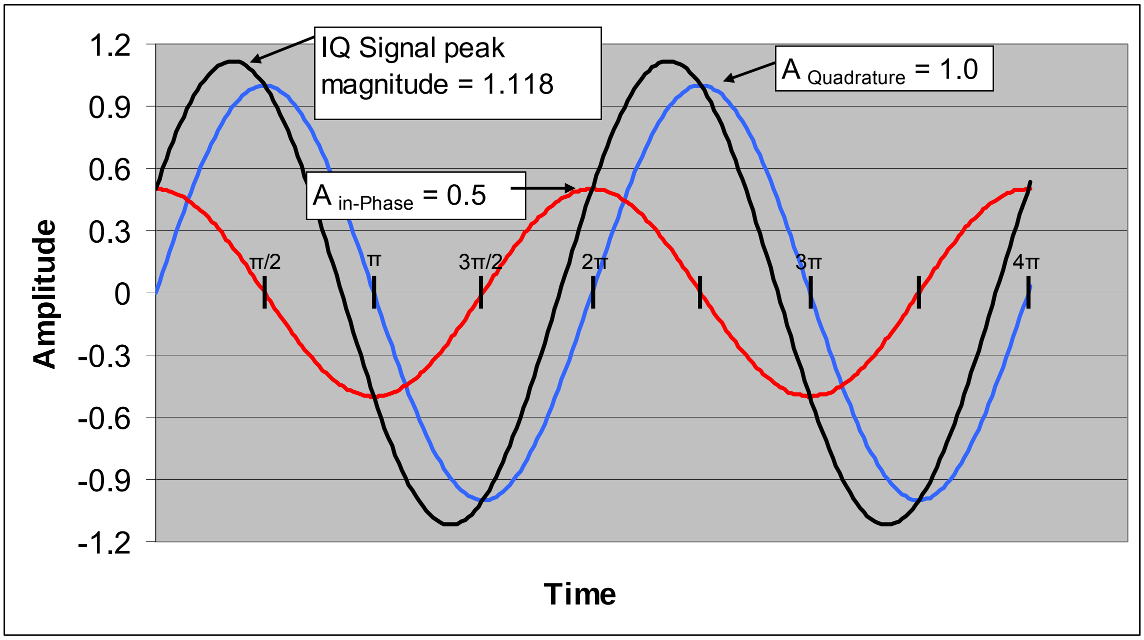


Figure 1.3.2: Time domain plot of the IQ signal and its In-phase and Quadrature component signals.

1.4 *Figure 1.7 shows how the magnitude and phase are calculated for an imaginary number that resides in the first quadrant of the plane (both real and imaginary components are positive). Show how we calculate the magnitude and phase of an imaginary number in the other quadrants.*

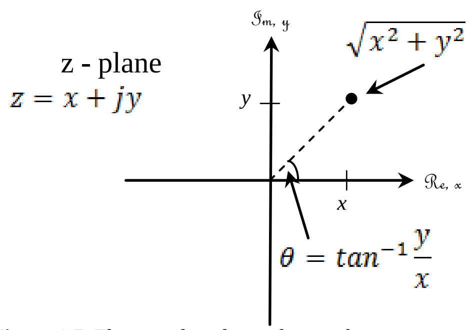
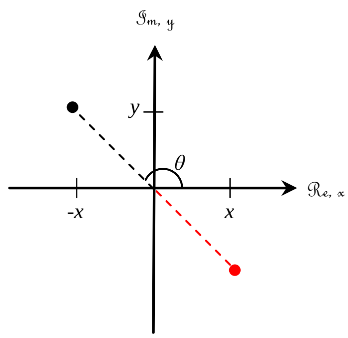


Figure 1.7 The complex plane, plotting the imaginary number

Let's begin by not thinking too hard on the solution and just figure out the magnitude and phase of an imaginary number that resides in Quadrant II.



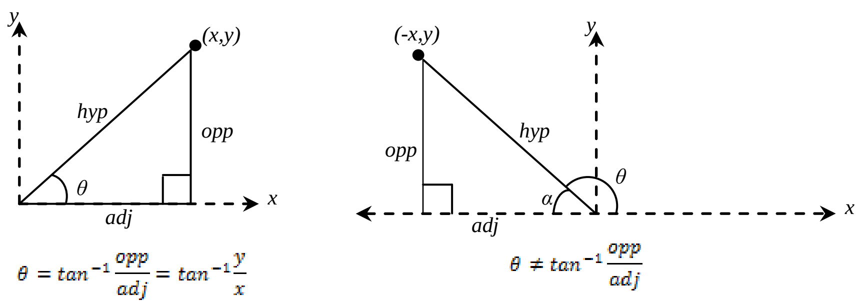
$$Let\ x = -1, and\ y = 1$$

$$|z| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2} \approx 0.707$$

$$\theta = tan^{-1}\frac{1}{-1} = \mathbf{-45°}$$

F-1 Plot and attempted angle determination for a point in the z-plane that lies in Quadrant II

Using the inverse tangent without thinking about what you are doing will give you the ***WRONG*** angle if the point resides in any Quadrants II, III, or IV. Let's revisit the definition of the inverse tangent:

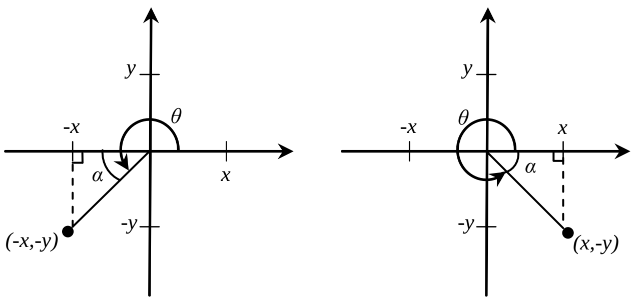


F-2 Inverse tangent function for two angles

You can see where the problem lies when you are trying to find the angle of a point in Quadrant II. The trigonometry functions are defined for a triangle with a right angle. You are no longer using the inverse tangent for $\boldsymbol{\theta}$, instead you are finding the inverse tangent of $\boldsymbol{\alpha}$ as seen in F-2. Fortunately, $\boldsymbol{\alpha}$ and $\boldsymbol{\theta}$ are supplementary and we can use the definition for supplementary angles to determine $\boldsymbol{\theta}$.

$$\alpha + \theta = \pi \;\therefore\; \theta = \pi - \alpha$$

Now that we have determined how to find the angle of a point that lies in Quadrant II. Let's plot points in Quadrants III and IV to find how to find determine their angle.



F-3 Inverse tangent function for two angles

Using F-2 and F-3 we can intuitively determine the correct angle by using the modified inverse tangent function below:

$$\theta = \left\{ \begin{array}{c} tan^{-1}\dfrac{y}{x} \leftrightarrow Quadrant\ I \\ \\ \pi - tan^{-1}\dfrac{y}{|x|} \leftrightarrow Quadrant\ II \\ \\ \pi + tan^{-1}\dfrac{|y|}{|x|} \leftrightarrow Quadrant\ III \\ \\ 2\pi - tan^{-1}\dfrac{|y|}{x} \leftrightarrow Quadrant\ IV \end{array} \right\}$$

1.5 If the output of a system occurs after the corresponding input to the system, is the phase shift positive or negative? Why? What does linear phase indicate?

Solution:

The phase shift is negative when the output of a system occurs after the corresponding input.

To illustrate "why", let us investigate the relation between two sinusoidal siganls $\sin(\omega t)$ & $\sin(\omega t-\theta)$. So when these two sinusoidal functions have the same value, $\omega t = \omega t' - \theta$ which leads to $t' = t + \dfrac{\theta}{\omega}$ meaning that the sinusoidal signal with negative phase shift occurs after the corresponding sinusoidal signal.

Why linear phase?

When a sinusoidal signal $x(t) = \sin(\omega_0 t)$ is passed into a system with transfer function of $H(\omega) = A(\omega)e^{j\theta(\omega)}$, and the output signal has the form of

$$y(t) = |A(\omega)|\sin(\omega_0 t + \theta(\omega_0)) = |A(\omega)|\sin\left(\omega_0\left(t + \frac{\theta(\omega_0)}{\omega_0}\right)\right).$$

Now, if the system has linear phase, which indicates $\theta(\omega_0) = k\omega_0$, the output can be rewritten as

$$y(t) = |A(\omega)|\sin(\omega_0(t+k))$$

This means the system only shifts the input signal in time and does not distort it.

1.6) **Using the SPICE files found at CMOSedu.com, verify, in the time-domain, the frequency response information seen in Fig. 1.10 for input frequencies of *f*=0 (DC), 1/4t_d, and 1/2t_d.**

Ans) The frequency response shown in Fig. 1.10 is that of a comb filter.

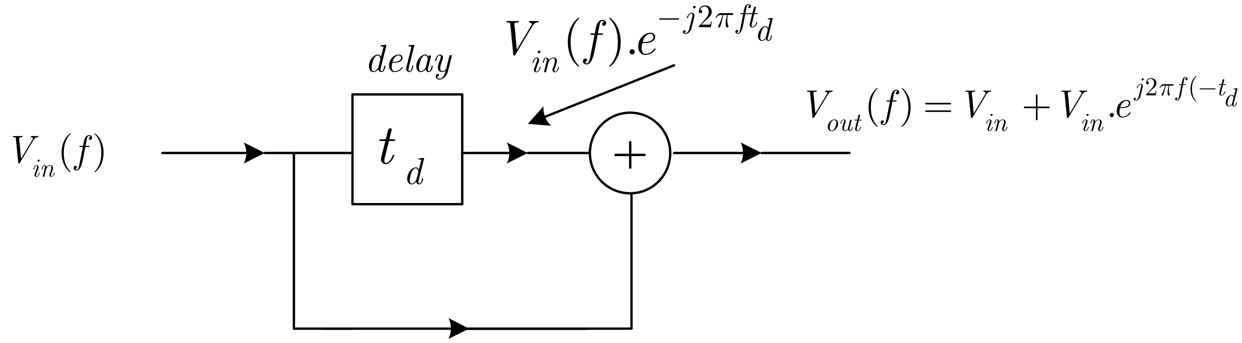


Figure 1: Block diagram of a comb filter

For the comb filter shown above, $V_{out} = V_{in} + V_{in}.e^{j.2\pi f.(-t_d)}$

The magnitude of the transfer function is found to be,

$$\left|\frac{V_{out}}{V_{in}}\right| = 2.\left|\cos \pi.f.t_d\right| \quad (1)$$

and the phase response is given by,

$$\angle\frac{V_{out}}{V_{in}} = \pi.(-t_d).f \text{ for } f<1/(2t_d). \quad (2)$$

Shown below are the plots of magnitude and phase response of the comb filter.

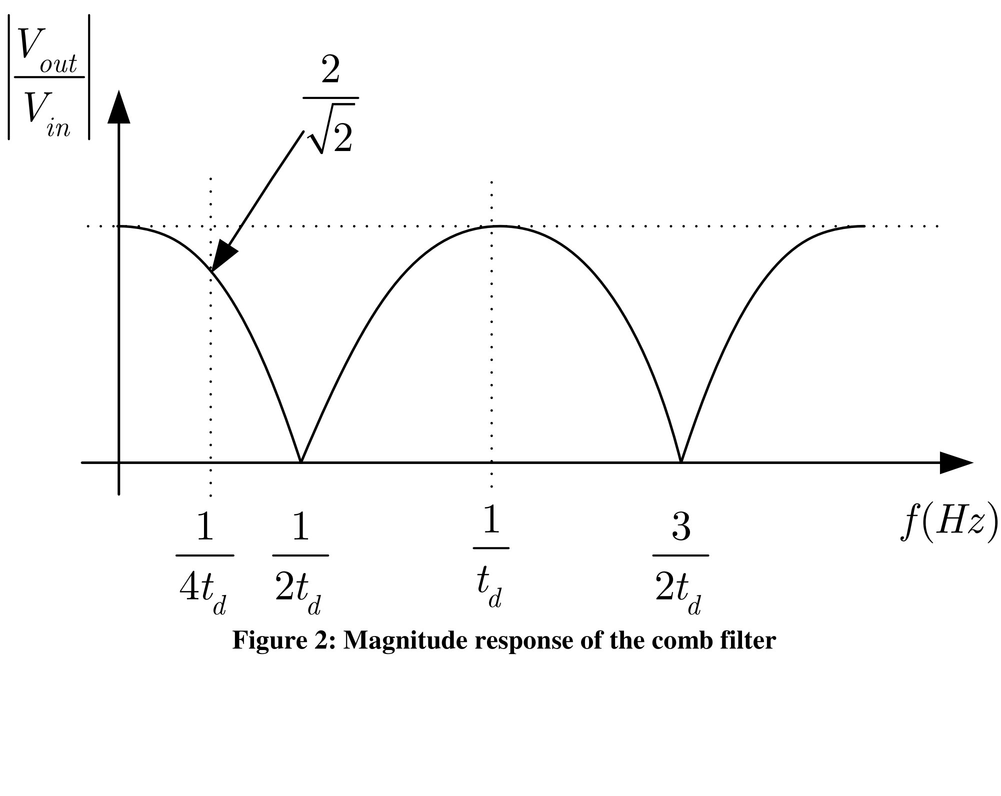


Figure 2: Magnitude response of the comb filter

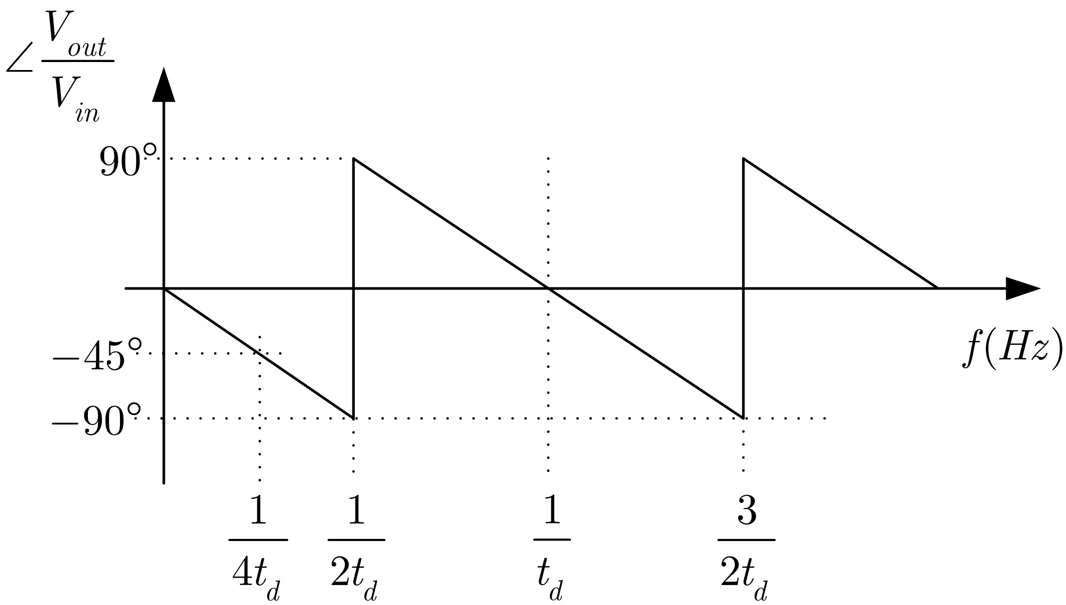


Figure 3: Phase response of a comb filter

The phase response of the comb filter is a linear response, indicating a constant delay through it. This is very useful for a distortion less filtering of the signals.

The above comb filter can be simulated using the circuit shown below, which simulates an analog comb filter. Notice here the value of the time delay through the transmission line is td = 5nsec.

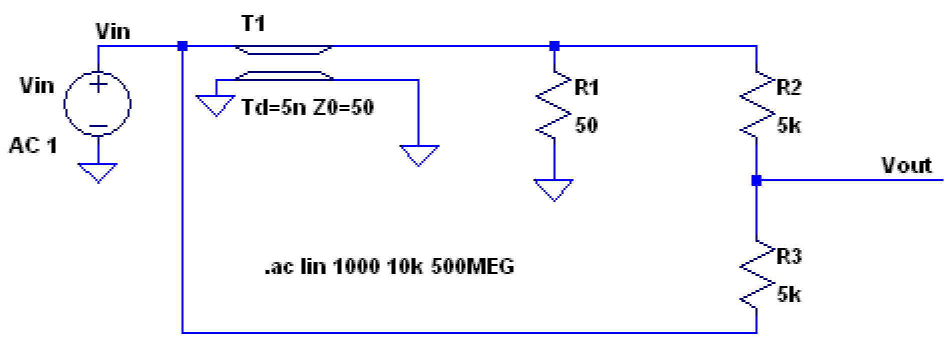


Figure 4: Implementation of an analog comb filter

Since the output voltage signal Vout is an average of the signal on the top and bottom transistors R2 and R3, the magnitude response is scaled down by 2 when compared to eq(1). Hence the magnitude response of the comb filter in fig.4 is,

$$\left|\frac{V_{out}}{V_{in}}\right| = \left|\cos \pi . f . t_d\right| \qquad (3)$$

Simulation results:

Input frequency $f = 0$ (DC signal)

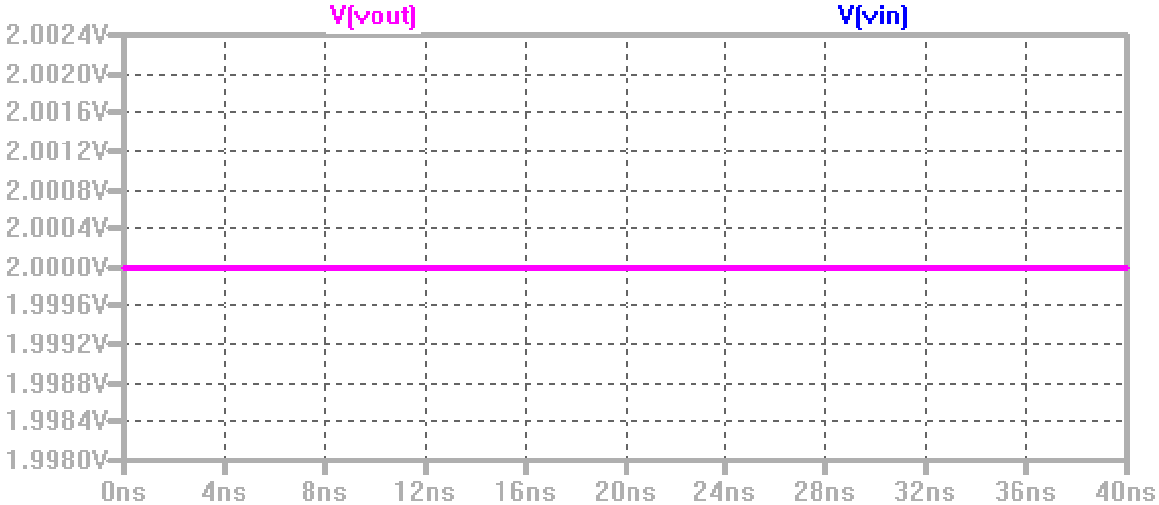


Figure 5: Time domain input and output (at DC) for the comb filter in fig. 4

The input signal is a DC voltage of amplitude 2 volts. Since a DC voltage cannot have any delay, it passes through the transmission line unaffected. The voltages on either nodes of R2 and R3 are at the same potential, so no current flows through both the resistors. Hence, the complete voltage Vin of 2volts gets transmitted to the output.

Input frequency $f = 1/4t_d$

Since the time delay for the given transmission line in the comb filter is 5nsec. The frequency of the input signal is equal to 50MHz, for a $f=1/4t_d$. From fig.2, we see at an input frequency of $1/4t_d$ the magnitude of the output signal is $2/\sqrt{2}$. The expected output of the analog comb filter implementation should be scaled down by 2 (Vout = $1/\sqrt{2}$) due to the presence of the two identical resistors at the output of the circuit. (See Eq.3)

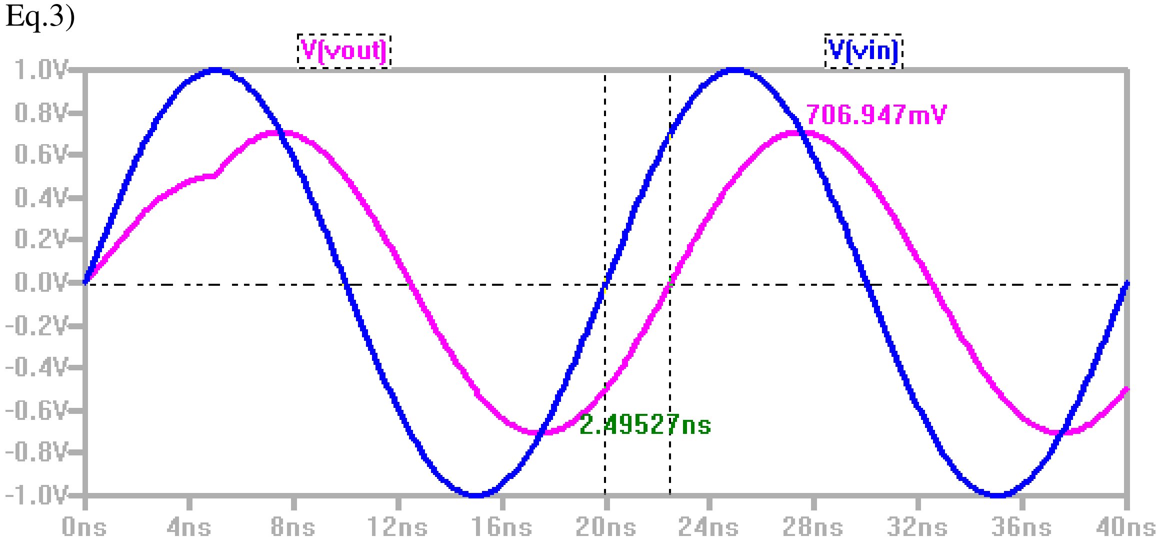


Figure 6: Time domain input and output (50 MHz) for the comb filter

As expected the magnitude of the output signal is $1/\sqrt{2} = 0.707$V. From fig.3, we see that for input signals of frequency equal to $1/4t_d$ the phase is -45°, which corresponds to a time lag of 2.5ns between the input and output signals, as expected the delay of Vout in the above plot matches the phase shift in fig.3.

Input frequency $f = 1/2t_d$ (100MHz)

From fig.2, we see that the comb filter attenuates the input signals at frequencies that are multiples of $1/2t_d$.

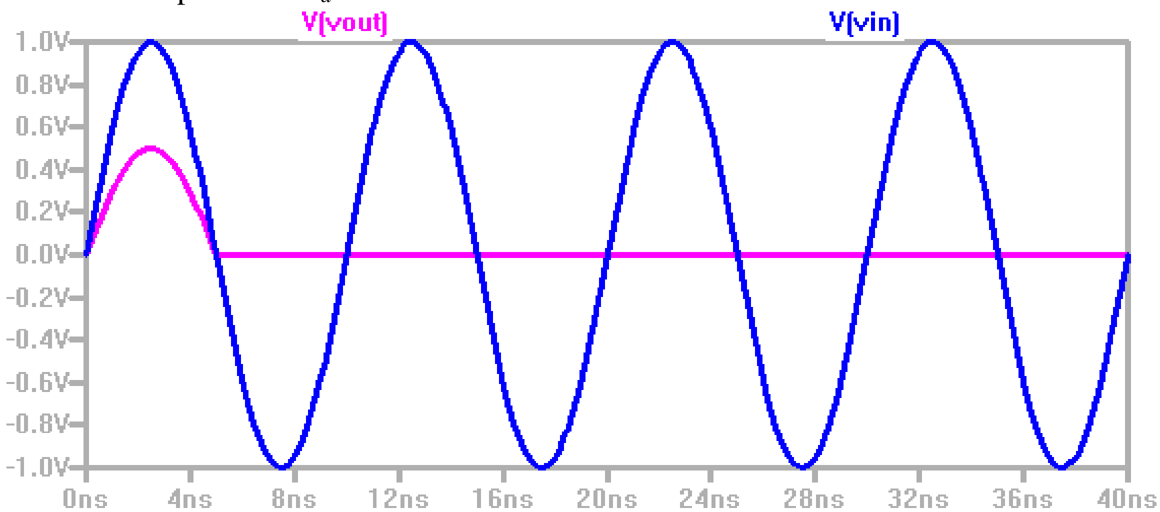


Figure 7: Time domain input and output (100 MHz) for the comb filter

Initially, due to start-up time of the circuit, the input gets passed on to the output, but once the start-up time is passed, the filter attenuates the input signal as seen in fig. 7.

For an input signal of frequency $f=1/2t_d$=100MHz, which corresponds to a time period of 10nsec for the input sinusoid. Due to the delay of 5nsec through the transmission line, if the sinusoidal voltage on the R2 resistor is at its peak, then the sinusoidal voltage on the R3 resistor will be at its valley. These two signals cancel out, giving rise to the attenuation of the input signal at the output. At this frequency, the phase shift is 180°, which means the input and output cancel each other out as expected from fig2 and fig.3.

This attenuating property of the comb filter is used in communication systems to isolate and prevent crosstalk between transmission channels.

Net list example for Fig.2

```
T1 Vin 0 N001 0 Td=5n Z0=50
R1 N001 0 50
R2 N001 Vout 5k
R3 Vout Vin 5k
Vin Vin 0 SINE(0 1 50Meg) AC 0
.tran 0 40ns 0
.end
```

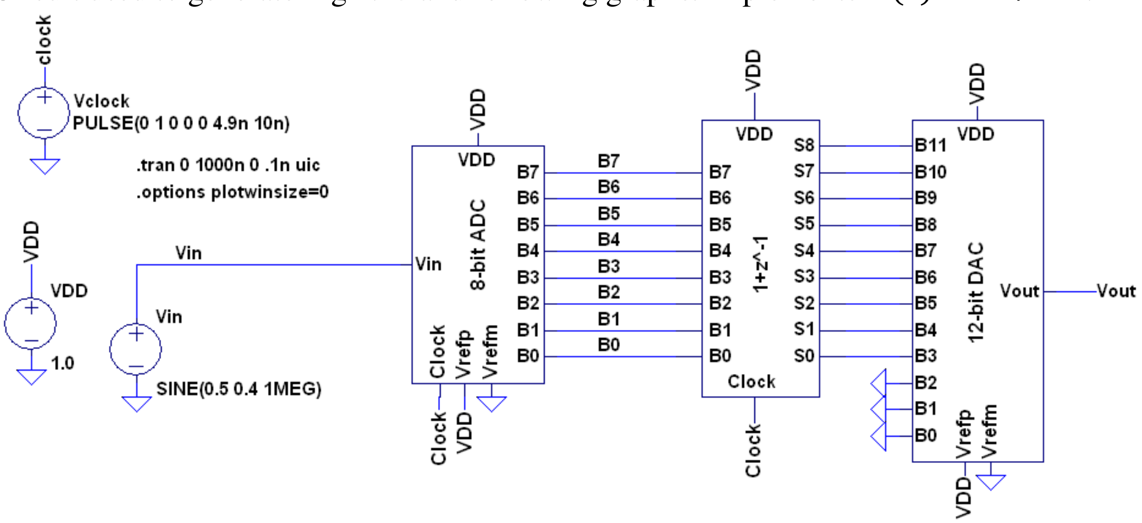
Reference
J. Baker, *CMOS Mixed Signal Circuit Design*, Second Edition, John Wiley and Sons, 2009. ISBN 978-0-470-29026-2

Jason Durand

Problem 1.7 – Using the spice files found at CMOSedu.com, verify, I the time domain, the frequency response information seen in Fig 1.17 for input frequencies of DC, $f_s/4$, and $f_s/2$.

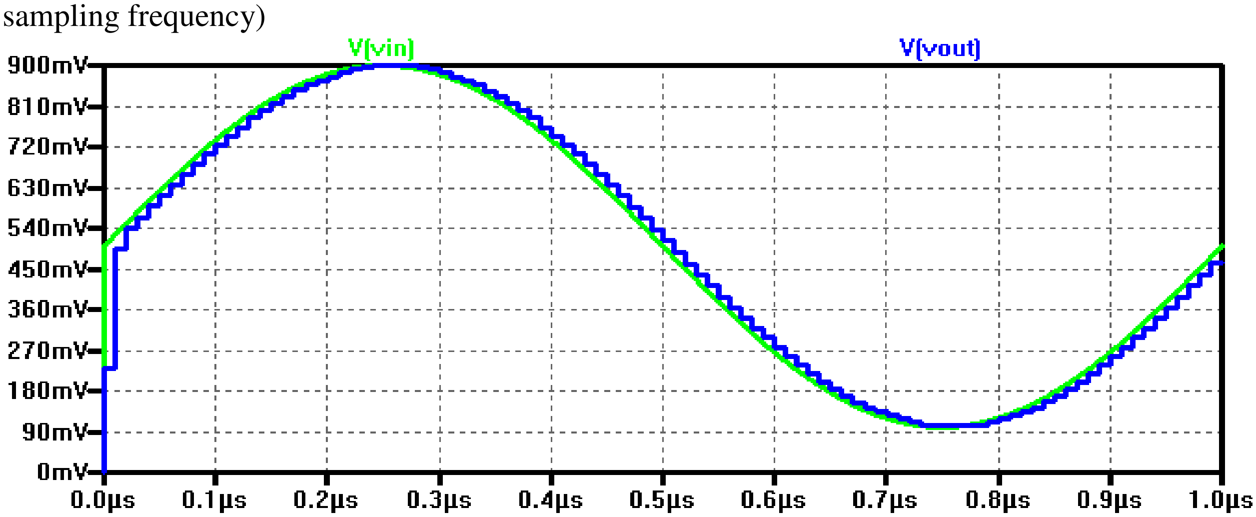
Fig 1.17 shows the magnitude and phase response of a comb filter/averager with a transfer function $H(z) = 1 + z^{-1}$. The magnitude response of the transfer function should be two at DC, $2/\sqrt{2}$ at $f_s/4$, and zero at $f_s/2$. The circuit made with ideal elements seen below scales the output magnitude by ½ to prevent the adder element from overflowing and wrapping around with a DC input.

Circuit used to generate Fig 1.17 and following graphs. Implements $H(z) = 1 + z^{-1}$.



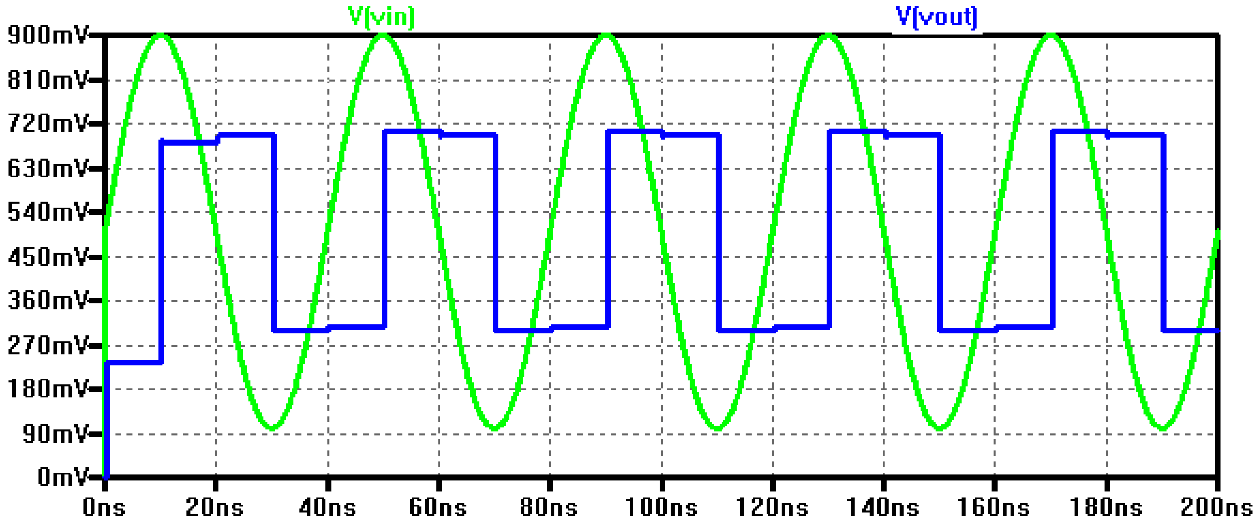
f_s = 100MHz, f_n = 50 MHz.

Time-domain response of circuit with (near) DC input (1MHz is sufficiently below the 100MHz sampling frequency)



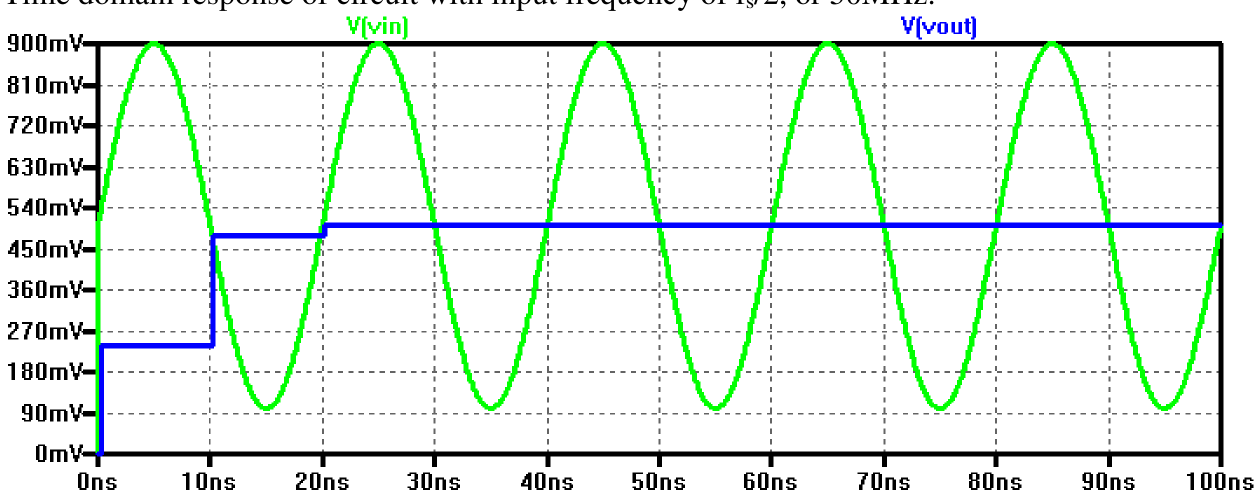
Notice that the magnitude response is unity as expected, and the signal is not delayed (zero phase response)

Time domain response of circuit with input frequency of $f_s/4$, or 25MHz.



The input amplitude, peak to peak, is 800mV, and the output amplitude peak to peak is much smaller than that, about 700mV – 300mV = 400mV. From Fig 1.17, the magnitude response should be $\sqrt{2}$, and remembering that it is scaled by ½ in the simulation, $\sqrt{2}/2$ = 0.707. 800mV scaled by 0.707 is 565mV, which is approximately the 400mV peak to peak displayed in the graph. Phase response is -45 degrees.

Time domain response of circuit with input frequency of $f_s/2$, or 50MHz.



As expected, the magnitude response of the signal at $f_s/2$ is zero. The phase response is meaningless with zero magnitude response, though it would be -90 degrees.

1.8. Plot the magnitude and phase frequency response of a discrete time system having the transfer function $\frac{1+z^{-1}}{z^{-2}}$. Next, show the location of this system's poles and zeros in the complex plane and verify, using the intuitive method discussed in Sec. 1.2.3, the gain and phase of the response match the frequency response plots when the input signal frequency is 0.

Sol:

The frequency response *H(f)* of a discrete time system can be determined by evaluating *H(z)* along the unit circle. The sampling frequency is denoted by *fs*.

Hence $$H(z)=\frac{1+z^{-1}}{z^{-2}} \text{ where } z=e^{j2\pi f/fs} \tag{1}$$

$$H(f)=\frac{1+e^{-j2\pi f/fs}}{e^{-j4\pi f/fs}} \tag{2}$$

Assume $x = -2\pi f / fs$

$$H(f)=\frac{1+\cos(x)+j\sin(x)}{\cos(2x)+j\sin(2x)} \tag{3}$$

$$H(f)=\frac{2\cos^2\frac{x}{2}+j2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos(2x)+j\sin(2x)} \tag{4}$$

$$H(f)=\frac{2\cos\frac{x}{2}(\cos\frac{x}{2}+j\sin\frac{x}{2})}{\big(\cos(2x)+j\sin(2x)\big)} \tag{5}$$

$$H(f)=\frac{2\cos\frac{x}{2}e^{jx/2}}{e^{j2x}} \tag{6}$$

$$\text{Note } H(f)=\frac{v_{out}}{v_{in}} \tag{7}$$

Since division of unit magnitude complex numbers $e^{jx/2}, e^{j2x}$ results in subtraction of phase of the individual complex numbers the system response is as follows.

Real part Imaginary part

$$\frac{v_{out}}{v_{in}}=2\cos\frac{x}{2}\cos\left(\frac{-3x}{2}\right)+j2\cos\frac{x}{2}\sin\left(\frac{-3x}{2}\right) \tag{8}$$

The magnitude response of the system is given by

$$\left|\frac{vout}{vin}\right| = \sqrt{\left(2\cos\frac{x}{2}\right)^2 \cos^2\frac{3x}{2} + \left(2\cos\frac{x}{2}\right)^2 \sin^2\frac{3x}{2}} \tag{9}$$

$$\left|\frac{vout}{vin}\right| = 2\left|\cos\frac{x}{2}\right| \tag{10}$$

$$\left|\frac{vout}{vin}\right| = 2\left|\cos\left(-\frac{2\pi f}{fs\times 2}\right)\right| \tag{11}$$

$$\left|\frac{vout}{vin}\right| = 2\left|\cos\frac{\pi f}{fs}\right| \tag{12}$$

Phase response is given by from pole-zero plot of *H(z)* in figure 1

$$\angle H(f) = \angle\text{of zero - }\angle\text{of pole} \tag{13}$$

Noting that there are two poles at 0, and a zero at -1

$$\angle\frac{vout}{vin} = (\angle\frac{\pi f}{fs}) - (\angle\frac{2\pi f}{fs}) - (\angle\frac{2\pi f}{fs}) \tag{14}$$

$$\angle\frac{vout}{vin} = -\frac{3\pi f}{fs}\text{ for } 0 \le f \le \frac{fs}{2} \tag{15}$$

The magnitude and phase response plotted over frequency $0 \le f \le 2fs$ using matlab is shown below. Note that the magnitude response has zeroes at periodic multiples of *fs/2*, and peaks at integral multiples of *fs*. The phase response is a linear function. Hence the delay through the filter is constant.

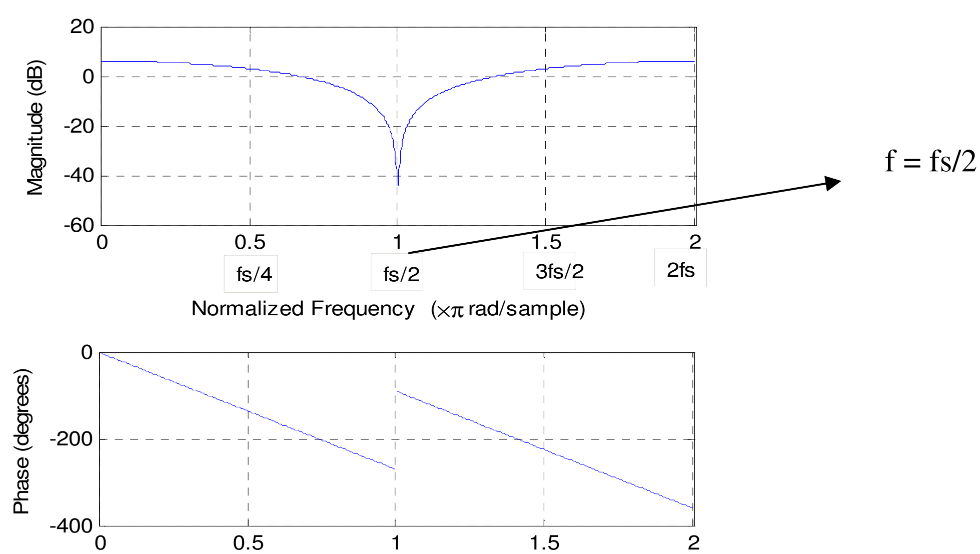


figure 1. Magnitude and Phase response of *H(f)*

The zeroes in the transfer function *H(z)* are the roots of the numerator given by $1 + z^{-1} = 0$. The root of this equation is -1. The poles are the roots of denominator given by $z^{-2} = 0$. The roots of this equation are 0, 0. Hence there is a zero at -1, and a doublet pole at zero.

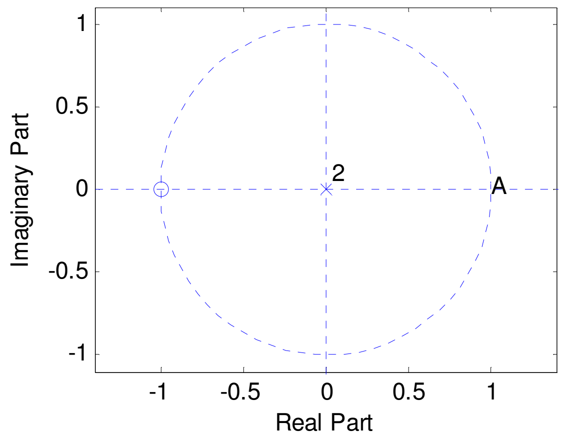


figure 2. Pole-zero plot of *H(z)*

The distance from point A to zero in the pole-zero plot is 2. The distance from point A in the pole-zero plot is 1.

$$\left|H(f)\right|_{f=0} = \frac{\text{distance from A to zero}}{\text{distance from A to pole}} \tag{16}$$

$$\left|H(f)\right|_{f=0} = \frac{2}{1} \tag{17}$$

The value of $\left|H(f)\right|_{f=0}$ obtained with intuitive method matches with actual value.

$$\angle H(f) = \angle\text{of zero} - \angle\text{of pole} \tag{18}$$

Since the angle made by pole and zero with x-axis is zero, the phase at DC is zero, which verifies the intuitive method.

1.9) For the 3 delay element comb filter seen in Fig. 1.25, repeat question 1.6 for input frequencies of 0, fs/6 and fs/3.

The circuit below represents a digital comb filter with a 3-element delay.

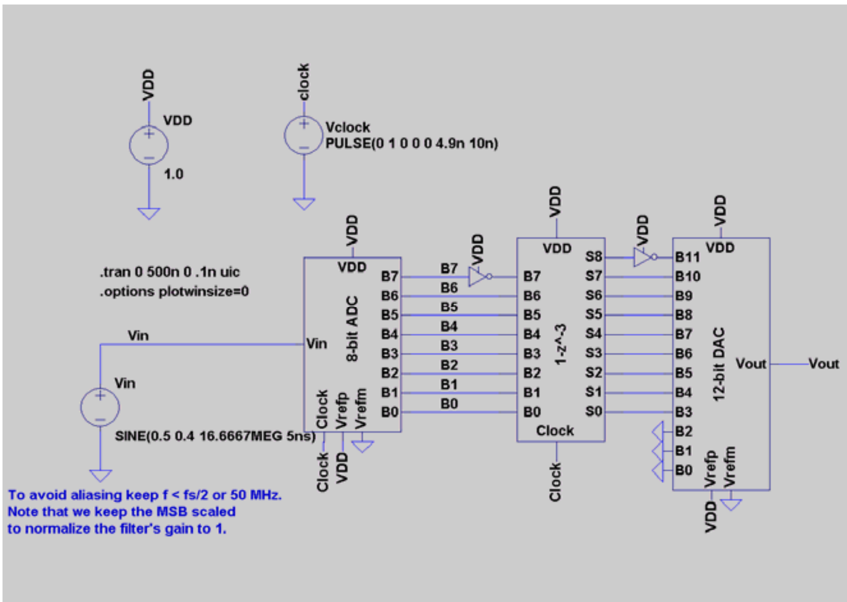
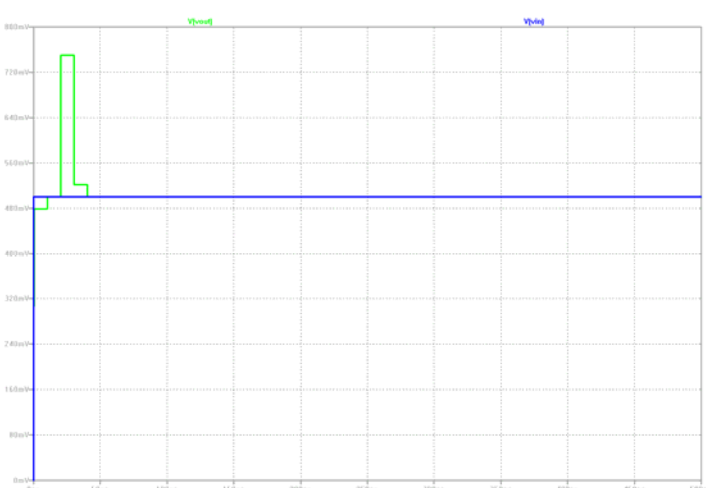


Fig.1 Digital Comb filter with 3-element delay

The frequency response of this filter has the zeroes at 1, $-1/2 + j\sqrt{3}/2$ and $-1/2 - j\sqrt{3}/2$. i.e, the transfer function is zero. The magnitude response for input frequencies that are multiples of $f_s/3$ is zero, and for odd multiples of $f_s/6$ [i.e., $(2n+1).f_s/6$], it's the maximum.

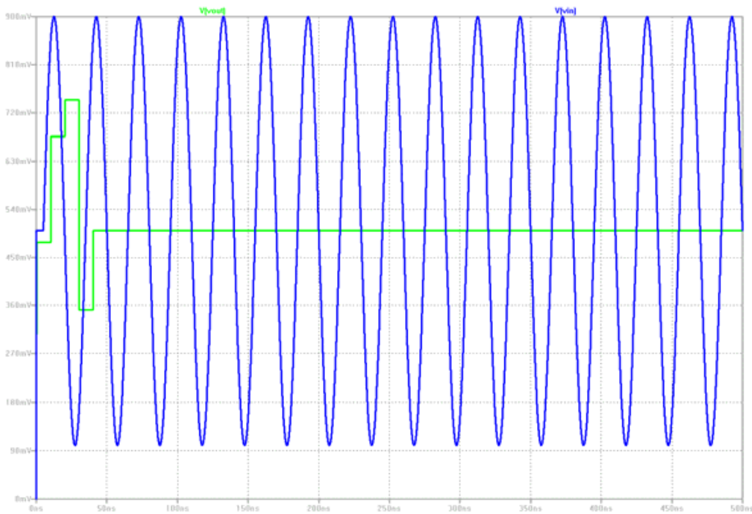
The following waveforms vary the input frequency, and represent the time-domain response:

i) input frequency f = DC = 0Hz



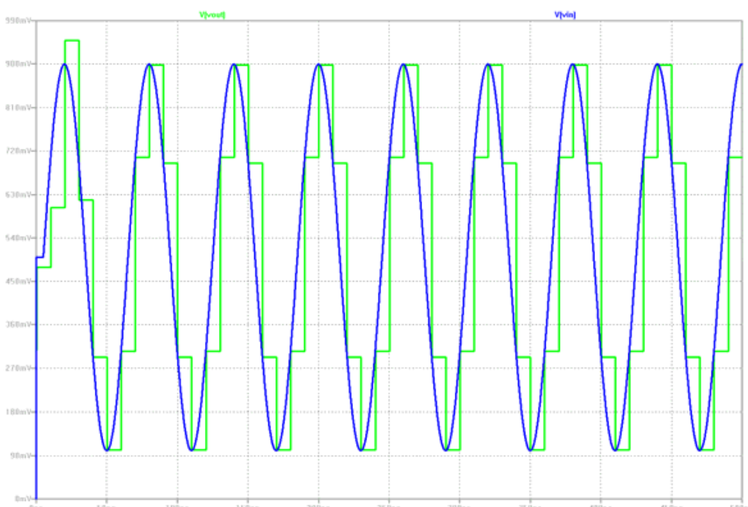
Notice that the output is identical to the input DC signal, after the initial setup period.

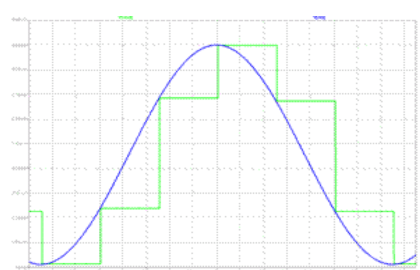
ii) input frequency f = fs/3 = 33.3333MHz



As expected from the frequency response, for an input frequency that is a multiple of fs/3, the output is zero.

iii) input frequency f = fs/6 = 16.6667MHz





For an input frequency that is multiple of fs/6 [and not of fs/3], the output tracks the input. The filter passes the input signal with a slight delay.

1.10 Show how to plot $1/(4+j3)$ in the complex plane. What is the magnitude and phase shift of this complex number?

Solution:

Any complex number z can be written in the form

$$z=a+jb \tag{1}$$

where a and b are the real and imaginary components of the complex number respectively. In the complex plane, the real component a is usually plotted along the x-axis as shown in Fig. 1. The imaginary component b is usually plotted along the y-axis. The magnitude of the complex number is

$$|z|=\sqrt{a^2+b^2}\ , \tag{2}$$

and the phase of the complex number is

$$\theta=\tan^{-1}\frac{b}{a}\ . \tag{3}$$

Fig. 1 shows the complex number $z=a+jb$ plotted in the complex plane.

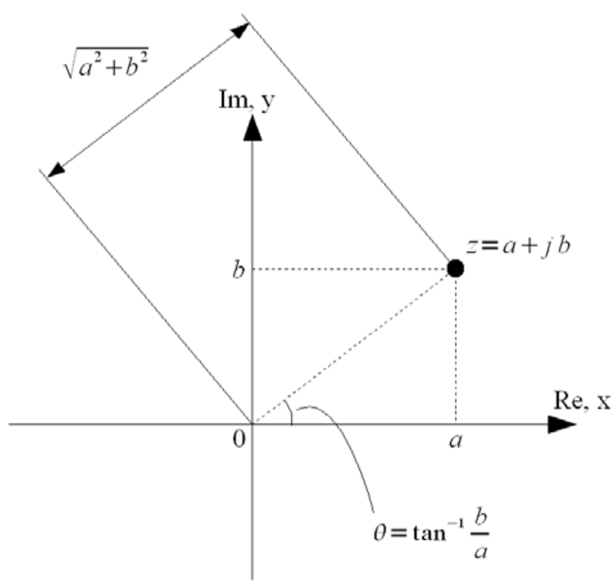


Figure 1 Complex number $z=a+jb$ and its magnitude and phase.

To plot a complex number in the form of

$$z=\frac{1}{c+jd} \tag{4}$$

we can manipulate it into the form as show in Eq. (1) by first multiplying both its numerator and denominator by the complex conjugate of $c+jd$, or $c-jd$. That is,

$$\begin{aligned}\frac{1}{c+jd}&=\frac{1}{c+jd}\cdot 1\\&=\frac{1}{c+jd}\cdot\frac{c-jd}{c-jd}\\&=\frac{c-jd}{c^2+d^2}\\&=\frac{c}{c^2+d^2}+j\frac{-d}{c^2+d^2}\end{aligned} \tag{5}$$

Comparing with Eq. (1) we see that $1/(c+jd)=a+jb$ when

$$\begin{aligned}a&=\frac{c}{c^2+d^2}\\b&=\frac{-d}{c^2+d^2}\end{aligned} \tag{6}$$

The magnitude of $1/(c+jd)$ is

$$\begin{aligned}|z|&=\sqrt{a^2+b^2}\\&=\sqrt{\left(\frac{c}{c^2+d^2}\right)^2+\left(\frac{-d}{c^2+d^2}\right)^2}\\&=\sqrt{\frac{c^2+d^2}{(c^2+d^2)^2}}\\&=\frac{1}{\sqrt{c^2+d^2}}\end{aligned} \tag{7}$$

and the phase is

$$\begin{aligned}\theta&=\tan^{-1}\frac{b}{a}\\&=\tan^{-1}\frac{-d/(c^2+d^2)}{c/(c^2+d^2)}\\&=\tan^{-1}(\frac{-d}{c})\\&=-\tan^{-1}(\frac{d}{c})\end{aligned} \tag{8}$$

The complex number $1/(4+j3)$ is in the form of Eq. (4) with $c=4$, and $d=3$. Substituting these values into Eq. (6) we have

$$a=\frac{4}{4^2+3^2}=0.16$$
$$b=\frac{-3}{4^2+3^2}=-0.12 \tag{9}$$

Thus

$$\frac{1}{4+j3}=0.16+j(-0.12) \tag{10}$$

Fig. 2 shows this complex number plotted in the complex plane.

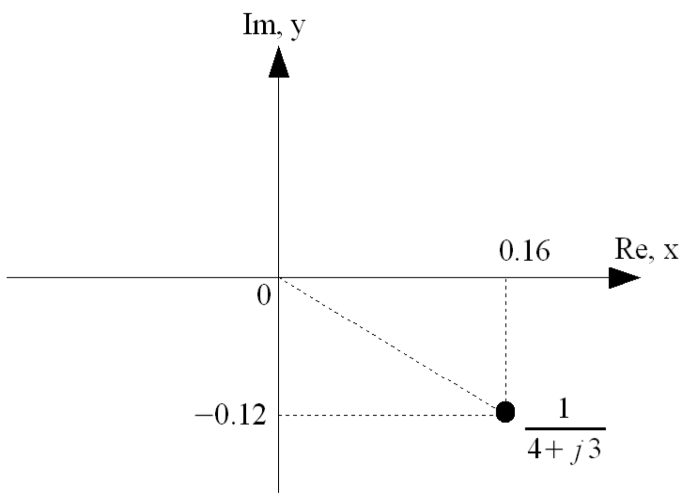


Figure 2 Plotting $1/(4+j3)$ in the complex plane.

The magnitude of 1/(4+j 3) can be calculated using the result of Eq. (7), or

$$\left|\frac{1}{4+j3}\right|=\frac{1}{\sqrt{4^2+3^2}}=0.2 \tag{11}$$

Its phase can be calculated using the result of Eq. (8), or

$$\theta=-\tan^{-1}\frac{3}{4}=-36.87^O \tag{12}$$

A complex number can be written in terms of its magnitude A and phase θ as

$$A\measuredangle\theta \tag{13}$$

For example, the complex number $1/(4+3j)$ can be written as

$$0.2\measuredangle-36.87^O \tag{14}$$

In many cases it is very convenient to see $1/(4+j3)$ as z_1/z_2 where

$$z_1=1+j0 \quad \rightarrow \quad 1\measuredangle 0^O \text{ and}$$

$$z_2=4+j3 \quad \rightarrow \quad 5\measuredangle 36.87^O$$

are two complex numbers. The magnitude of $z=z_1/z_2$ is

$$|z|=\frac{|z_1|}{|z_2|} \tag{15}$$

The phase of $z=z_1/z_2$ is

$$\theta=\theta_1-\theta_2 \tag{16}$$

where θ_1 and θ_2 are the phases of z_1 and z_2 respectively.

Thus for $z=1/(4+j3)$ we have

$$z=\frac{1}{4+j3} \rightarrow \frac{|z_1|}{|z_2|}\measuredangle(\theta_1-\theta_2)=\frac{1}{5}\measuredangle(0^O-36.87^O)=0.2\measuredangle-36.87^O,$$

which are the same results as we show in Eqs. (11) and (12) .

Lincoln Bollschweiler

1.11 Determine the z-domain representation of the circuit seen in Fig. 1.30. Also, plot the frequency response, both magnitude and phase, and the location of poles and zeros for this system.

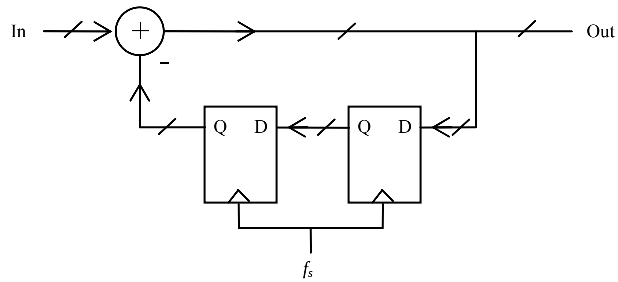
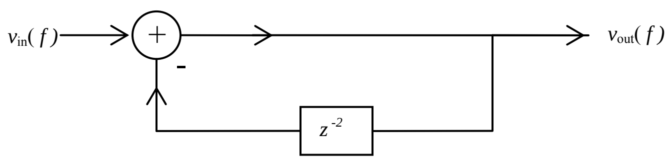


Figure 1.30

Each flip flop represents one delay of $1/f_s$ for $2T_s$ total delay. Since $z^{-1} = e^{j2\pi f\cdot(-T_s)}$ (Eq. 1.37), representing a delay of T_s, then a delay of $2T_s$ can be represented by $z^{-2} = e^{j2\pi f\cdot(-2T_s)}$. Now we can redraw the circuit for with the z-domain representation of the digital delay.



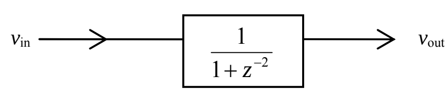
We can write the transfer function as

$$v_{out} = v_{in} - v_{out}\cdot z^{-2},$$

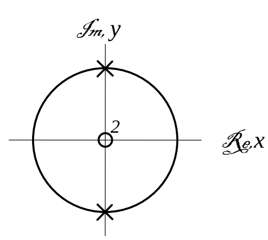
$$v_{out}\left(1+z^{-2}\right) = v_{in},$$

$$H(z) = \frac{v_{out}}{v_{in}} = \frac{1}{1+z^{-2}} = \frac{z^2}{z^2+1} = \frac{z^2}{(z+j)(z-j)}.$$

The z-domain circuit can be simplified to:



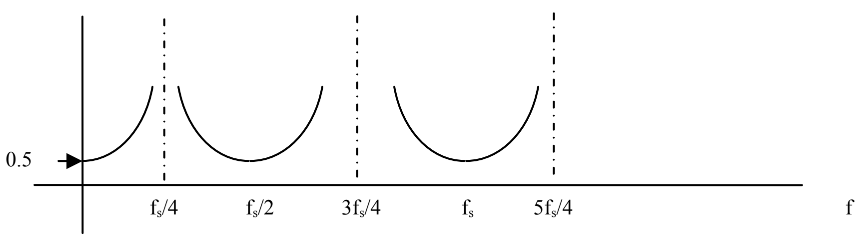
The poles and zeros can be plotted in the z-plane as follows:



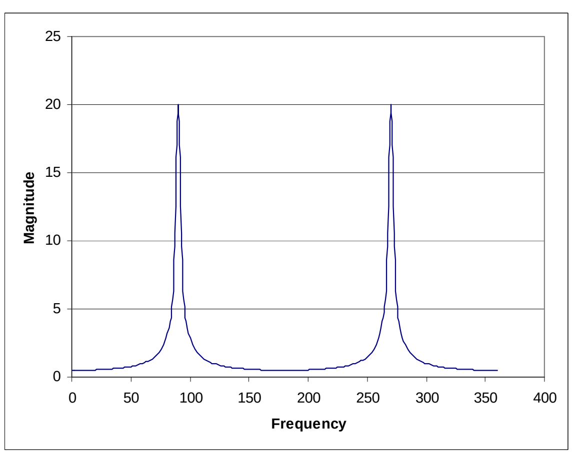
We can use the z-plane plot to determine the magnitude and phase response for the circuit. We know that

$$|H(z)| = \frac{\text{distance to zero}}{\text{distance to pole}} \text{ and } \angle H(z) = \angle \text{ of the zero } - \angle \text{ of the pole.}$$

Following the method used in Example 1.3 on page 17, from examination we can see that the zeros each always have a distance of 1 from any point on the unit circle. We can find the minimum magnitude by finding the largest distance for the poles. This occurs when they are equal distance from a point on the circle. This occurs at f = 0, $f_s/2$, f_s, etc. and can be found to be $\sqrt{2}$ for each pole. Using the fact that magnitude is the product of all individual magnitudes, the minimum magnitude is then $|H(z)| = \frac{1 \cdot 1}{\sqrt{2 \cdot \sqrt{2}}} = \frac{1}{2} = 0.5$. This can be verified by trigonometry. If x and y represent the distances from the poles to a point on the circle then $\sin\theta = x/2$ and $\sin(90 - \theta) = y/2$. One could take the derivative of $x \cdot y$ to find local minima and maxima. Doing this, and setting this to zero to find the maxima and minima reveals that these occur at 0°, 90°, 180°, and 270°, meaning that 90° and 270° are the points for the largest magnitude of the poles, yielding the smallest overall magnitude (0.5). The maximum magnitude occurs at points on the circle that coincide with one of the poles. These points occur at f = $f_s/4$, $3f_s/4$, $5f_s/4$, etc. Here the magnitude is $|H(z)| = \frac{1}{0} = \infty$. The resulting magnitude frequency response can be seen here.



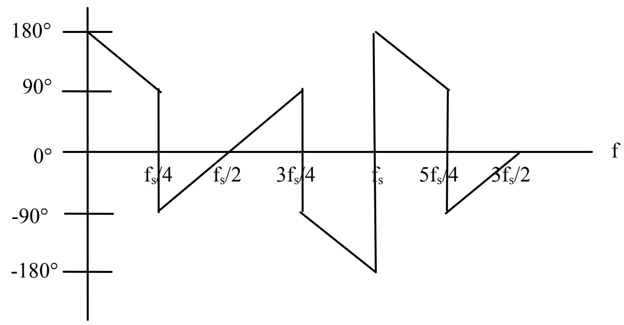
A better plot was generated with a spreadsheet, which calculated the magnitudes, *substituting 20 for infinity* at $f_s/4$ and $3f_s/4$. The plot assumes $f_s = 360$.



Following the procedure for determining phase response, also from Example 1.3, one finds the following phase responses, where Phase = (Zero 1 + Zero 2) – (Pole 1 + Pole 2):

Frequency	**Zero 1**	**Zero 2**	**Pole 1**	**Pole 2**	**Phase**
0	0°	0°	135°	45°	180°
$f_s/4_{\text{from below}}$	90°	90°	0°	90°	90°
$f_s/4_{\text{from above}}$	90°	90°	180°	90°	-90°
$f_s/2$	180°	180°	225°	135°	0°
$3f_s/4_{\text{from below}}$	270°	270°	270°	180°	90°
$3f_s/4_{\text{from above}}$	270°	270°	270°	0°	-90°

The phase frequency response is plotted here.



The circuit seen in this problem is known as an $f_s/4$ resonator. A simulation to verify these frequency responses will not be performed on this circuit. The discontinuities in magnitude (and phase) at $f_s/4$, $3f_s/4$, $5f_s/4$ …, require special care to be taken in simulation so that the simulator stops and starts calculating just prior to and just after these discontinuity locations. This technique and additional circuit components which aid the simulation of the $f_s/4$ resonator are developed in chapter 4. If further study on this circuit is desired, refer to Fig. 4.23 and the discussion surrounding it.

Q1.12 Repeat question 1.11, and sketch the resulting circuit, if a delay is added to the forward path of the circuit seen in Fig. 1.30.

Sol. Figure 1.30 shows the circuit given in question 1.11

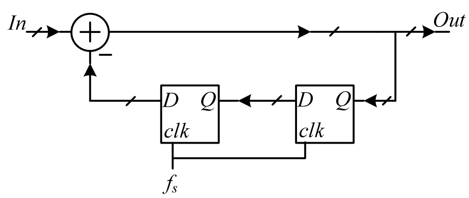


Fig. 1.30

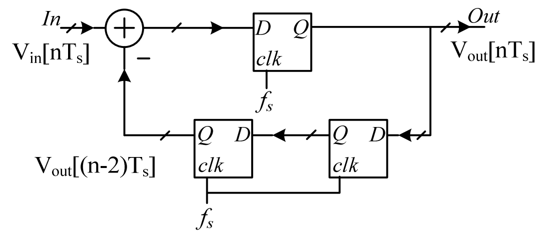


Fig. 1.1 Adding delay in forward path

Looking at the Figure 1.1 we see the system with added delay in the forward path. In terms of complex plane *z*, where $z = e^{j2\pi f/f_s}$ and f_s, clock frequency, the resulting circuit in *frequency* domain is shown below,

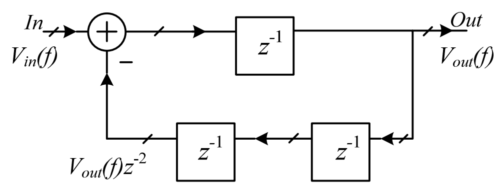


Fig. 1.2 circuit representation in terms of (z)

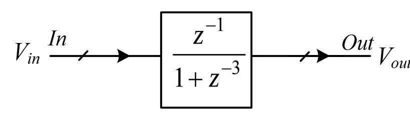


Fig.1.3 Transfer function of system

Thus looking at Figure 1.1 and 1.2 the equation in discrete time and frequency domain can be written as equation 1.1 and 1.2

$$Vout\left[nT_s\right] = Vin\left[(n-1)T_s\right] - Vout\left[(n-3)T_s\right] \tag{1.1}$$

$$Vout(f) = Vin(f)z^{-1} - Vout(f)z^{-3} \tag{1.2}$$

simplifying the equation further;

$$Vout\left(1+z^{-3}\right) = Vin\left(z^{-1}\right) \tag{1.3}$$

$$\frac{Vout}{Vin} = H(z) = \frac{z^{-1}}{1+z^{-3}} \quad \text{or} \quad H(f) = \frac{z^2}{1+z^3} \quad ;\, where\ \ z = e^{j2\pi f/fs} \tag{1.4}$$

Solving for magnitude of transfer function *H (f)*;

$$\left|H(f)\right| = \left|\frac{z^2}{1+z^3}\right| = \left|\frac{e^{2(j2\pi f/f_s)}}{1+e^{3(j2\pi f/f_s)}}\right| \text{ by complex number properties } \left|e^{\pm j\theta}\right| = 1 \text{ for any } \theta \text{ and } \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\left|H(f)\right| = \frac{\left|e^{j4\pi f/f_s}\right|}{\left|1+e^{j6\pi f/f_s}\right|} = \frac{1}{\left|1+\cos(6\pi f/f_s) + j\sin(6\pi f/f_s)\right|} \tag{1.5}$$

Further

$$\left|H(f)\right| = \frac{1}{\sqrt{\left(1+\cos(6\pi f/f_s)\right)^2 + \left(\sin(6\pi f/f_s)\right)^2}} = \frac{1}{2\left|\cos(3\pi f/f_s)\right|} \tag{1.6}$$

Thus $\boxed{\left|H(f)\right| = \frac{1}{2\left|\cos(3\pi f / f_s)\right|}}$ is the magnitude response of the system, we can see that based on the equation function will never go to zero because of 0<|cosine (θ)| <1 for all theta. Thus |*H (f)*| will be minimum when $\left|\cos(3\pi f / f_s)\right| = 1$ or in general $3\pi f / f_s = n\pi$ which leads to the solution $f = \frac{n}{3} f_s$ where n=0,1,2,3...... .At these frequencies the gain of system is half or 0.5.

For example |*H (f)*|= 0.5; for n= 0, *f* = 0 and cos (0) =1,

|*H (f)*|= 0.5; for n=1, $f = \frac{1}{3} f_s$ and cos (π) =1 so on and so forth.

Another information received from |*H (f)*| is about the position of poles of the system or points where the gain or |*H (f)*| of system tends infinity. All the poles occur at the point where $\cos(3\pi f / f_s) = 0$ or in general $3\pi f / f_s = (2n+1)\pi / 2$ which leads to the solution $f = \frac{(2n+1)}{6} f_s$ where n=0,1,2,3... from this we can easily evaluate the position of the poles. As seen in pole-zero plot (fig. 1.4) value of first three poles will come from n= 0,1 and 2 i.e. $f = f_s/6, f = f_s/2$ *and* $f = 5f_s/6$ respectively. The corresponding location of poles in z plane can be figured out by substituting these values of *f* in $e^{j2\pi f / f_s}$ which gives the location of the poles as:

$p1 = \frac{1}{2} + j\frac{\sqrt{3}}{2}$ at $f = f_s/6$; $p2 = -1$ at $f = f_s/2$; $p3 = \frac{1}{2} - j\frac{\sqrt{3}}{2}$ at $f = 5f_s/6$

Solving for phase of *H (f)*;

$$\angle H(f) = \frac{\angle\left(e^{j4\pi f / f_s}\right)}{\angle\left(1 + e^{j6\pi f / f_s}\right)} = 4\pi f / f_s - \tan^{-1}\left(\frac{\sin(6\pi f / f_s)}{1 + \cos(6\pi f / f_s)}\right) \tag{1.7}$$

leads to the solution

$$\angle H(f) = 4\pi f / f_s - \tan^{-1}\left[\tan\left(3\pi f / f_s\right)\right] = \pi f / f_s \tag{1.8}$$

Thus phase response is given by $\boxed{\angle H(f) = \pi f / f_s \;\; ; \text{for } 0 < f < f_s/2}$ which is linear in nature. Figure 1.4 shows the pole-zero plot for the given transfer function and Fig. 1.5 shows the magnitude and phase response for the same.

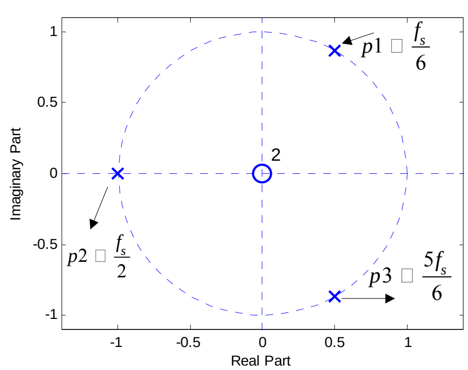


Fig. 1.4 z-plane showing poles and zeroes

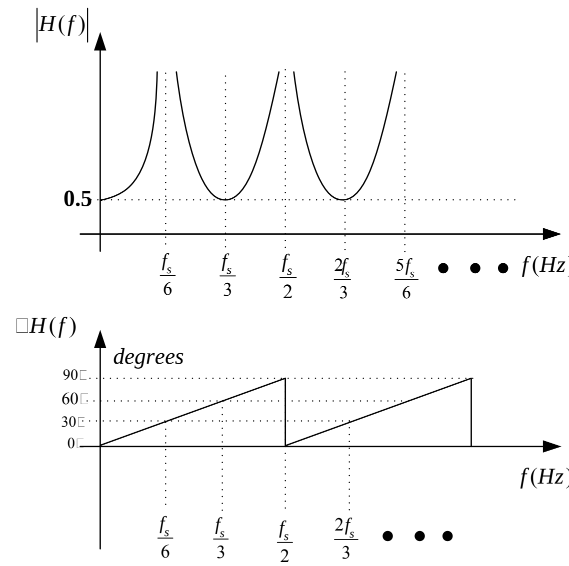


Fig. 1.5 Magnitude and Phase response of H (f)

There are 3 poles and two zeroes in the system, where location of poles are as stated above, the two zeroes are situated at origin. From phase and frequency response system properties are that of resonator.

Problem 1.13

Determine the exponential Fourier series representation for the squarewave seen in Fig. 1.29 if it is centered around ground.

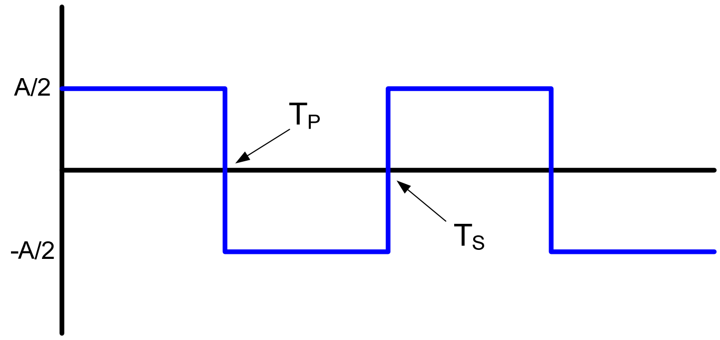


Figure 1.13.1: Waveform from book figure 1.29 centered around ground. This is the Signal that will be represented by the exponential Fourier series below.

I will begin by copying the relevant equations from the book 1.68:

$$c_n = \frac{1}{T}\int_{t}^{t+T} g(t) * e^{-j2\pi n f t}\,dt \qquad 1.13.1$$

The function g(t) in equation 1.13.1 represents the signal that we will convert into the Fourier Series Representation. It is a piecewise function defined over a single period as follows:

$$g(t) = \begin{cases} A/2, & 0 < t < T_P \\ -A/2, & T_P < t < T_S \end{cases} \qquad 1.13.2$$

In solving for the coefficients, the problem definition takes care of the 0^{th} order value of c_n by shifting the squarewave from the book figure 1.29 to be centered around ground. If the duty cycle of the squarewave is 50%, then the 0^{th} order value of c_n is 0. On the other hand, if the duty cycle is something other than 50%, than the dc component can be calculated simply as the average value of the signal over 1 period as follows:

$$c_0 = \frac{1}{T_S}\int_{0}^{T_S} g(t)\,dt \qquad 1.13.3$$

This value can be obtained geometrically as follows:

$$c_0 = \frac{A}{2} * T_P - \frac{A}{2} * (T_S - T_P) \qquad 1.13.4$$

Continuing the analysis of the non-zero coefficients we write:

$$c_n = \frac{1}{T_S}\left[\frac{A}{2}\int_0^{T_P} e^{-j2\pi n f_s t}dt - \frac{A}{2}\int_{T_P}^{T_S} e^{-j2\pi n f_s t}dt\right] \qquad 1.13.5$$

$$c_n = \frac{1}{T_S}\left[\frac{-A}{j4\pi n f_s}(e^{-j2\pi n f_s T_P} - 1) + \frac{A}{j4\pi n f_s}(e^{-j2\pi n f_S T_S} - e^{-j2\pi n f_S T_P})\right] \qquad 1.13.6$$

$$c_n = \frac{-Aj}{4\pi n}(1 - e^{-j2\pi n f_s T_P}) - \frac{Aj}{4\pi n}(e^{-j2\pi n f_S T_S} - e^{-j2\pi n f_S T_P}) \qquad 1.13.7$$

$$c_n = \frac{-Aj}{4\pi n}(1 - 2e^{-j2\pi n f_s T_P} + e^{-j2\pi n}) \qquad 1.13.8$$

Now let us consider only the expression in parenthesis, and re-write the exponential terms as separate real and imaginary components:

$$(1 - 2e^{-j2\pi n f_s T_P} + e^{-j2\pi n}) = 1 - 2\cos(2\pi n f_s T_P) + j2\sin(2\pi n f_s T_P) + \cos(2\pi n) - j\sin(2\pi n) \qquad 1.13.9$$

Note that the value of cos(2πn) will evaluate to 1 for every integer value n. Similarly sin(2πn) will evaluate to 0 for every integer value n. Therefore:

$$c_n = \frac{-Aj}{4\pi n} * [2 - 2\cos(2\pi n f_s T_P) + j2\sin(2\pi n f_s T_P)] \qquad 1.13.10$$

$$c_n = \frac{A}{2\pi n}\sin(2\pi n f_s T_P) + \frac{-Aj}{2\pi n}[1 - \cos(2\pi n f_s T_P)] \qquad 1.13.11$$

Looking at the solution, we can see the real and imaginary components. From these components we are able to calculate the magnitude and phase as described using the following equation for magnitude:

$$Magnitude = \sqrt{\{\mathrm{Re}\}^2 + \{\mathrm{Im}\}^2} \qquad 1.13.12$$

Also, with a positive Real component, and a negative Imaginary component, we expect the phase to exist in quadrant IV. Therefore we use the following equation to calculate the phase angle:

$$Phase = 2\pi - \tan^{-1}\frac{|\mathrm{Im}|}{\mathrm{Re}} \qquad 1.13.13$$

Now we are ready to use equations 1.13.12 and 1.13.13 to evaluate c_n into magnitude and phase values:

$$Magnitude = \sqrt{\left(\frac{A}{2\pi n}\sin(2\pi n f_S T_P)\right)^2 + \left(\frac{A}{2\pi n}\cos(2\pi n f_S T_P) - \frac{A}{2\pi n}\right)^2} \qquad 1.13.14$$

$$Magnitude = \frac{A}{2\pi n} * \sqrt{2 - 2\cos(2\pi n f_S T_P)} \qquad 1.13.15$$

$$Magnitude = \frac{A}{2\pi n} * \sqrt{4\sin^2(\pi n f_S T_P)} \qquad 1.13.16$$

$$Magnitude = \frac{A}{\pi n} * |\sin(\pi n f_S T_P)| \qquad 1.13.17$$

Phase calculations are as follows:

$$phase = 2\pi - \tan^{-1}\left(\frac{|\mathrm{Im}|}{\mathrm{Re}}\right) = 2\pi - \tan^{-1}\left(\frac{2\sin^2(\pi n f_S T_P)}{2\sin(\pi n f_S T_P)\cos(\pi n f_S T_P)}\right) \qquad 1.13.18$$

$$phase = 2\pi - \tan^{-1}\left[\tan(\pi n f_S T_P)\right] = 2\pi - \pi n f_S T_P \qquad 1.13.19$$

Equations 1.13.17 and 1.13.19 indicates c_n coefficients and phases for all values of ‘n’ not equal to 0. Note that for the case where the duty cycle is 50%, we can simplify things. Using:

$$T_P = \frac{1}{2}T_S = \frac{1}{2f_s} \qquad 1.13.20$$

We can write the magnitude as:

$$Magnitude = \frac{A}{\pi n} * \left| \sin\left(\frac{\pi n}{2}\right) \right| \qquad 1.13.21$$

And the phase at 50% duty cycle is:

$$phase = 2\pi - \frac{\pi n}{2} \qquad 1.13.22$$

1.14 *Determine the exponential Fourier series representation for the square-wave seen in Fig 1.29 for the general case where $T_p \neq T_s/2$.*

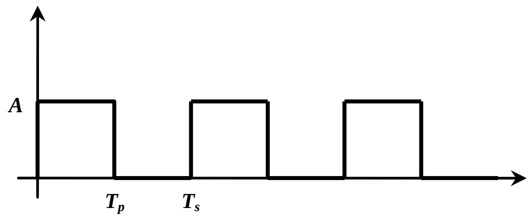
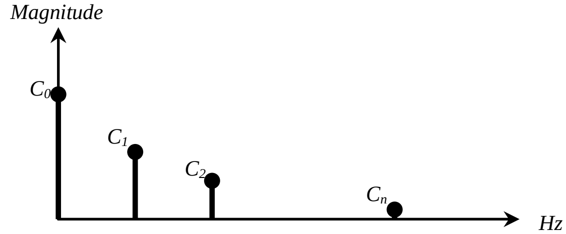


Figure 1.29 Representing a square-wave using exponential Fourier Series.

Figure 1.29 shows what we are used to seeing for a clock signal on an oscilloscope. Question 1.14 is being asked so that we understand what we are seeing when we plug a square-wave into a spectrum analyzer. Exponential Fourier series representation of periodic signals is used to describe what we expect to see on the display of a frequency analyzer. Let's begin our solution on page 22:

$$g(t) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{j2\pi nft}$$

What we see on the display of a frequency analyzer (when viewing a periodic signal) is *g(t)* and is defined by the equation above. Intuitively, we can guess that there will be an exponential decaying discrete signal in the frequency domain:



F-1 Intuitive plot of a periodic signal given the exponential Fourier Series representation

Let's see if we can prove F-1. The problem is asking for the magnitude of the *c* terms in the figure above. To find the magnitude of the coefficients let's use math to determine the magnitude:

$$c_n = \frac{1}{T}\int_{t}^{t+T} g(t) \cdot e^{-j2\pi nft}\, dt$$

Using this equation above noting that integration is the area under the curve. For the square-wave in figure 1.29 the area under the curve is zero for *g(t)* when *t* is between T_p and T_s.

$$c_n = \frac{1}{T_s}\int_{0}^{T_p} A \cdot e^{-j2\pi nf_s t}\, dt = \frac{A}{-j2\pi n} \cdot \left(e^{-j2\pi nf_sT_p} - 1\right)$$

Before we go further, please understand that we are going to use ***MATH*** to describe a periodic signal in the frequency domain so that we understand (among other things) what we are viewing when we use a spectrum analyzer.

There is a problem finding c_0 because both the numerator and denominator of the coefficient go to zero and we need to use l'Hospital's rule (differentiate the numerator and denominator until you get a valid value) to find c_0.

$$c_0 = \frac{AT_p}{T_s}$$

Returning to the frequency analyzer display, c_0 would be the peak of the fundamental spike and have the largest magnitude. To find the other coefficients let's return to the Fourier Series, and factor out j (remember $j = \sqrt{-1}$), and let $f_s = 1/T_s$:

$$c_n = \frac{A}{-j2\pi n} \cdot \left(e^{-j2\pi n f_s T_p} - 1\right) = \frac{-Aj}{2\pi n} \cdot \left(1 - e^{-j2\pi n \frac{T_p}{T_s}}\right)$$

Using Euler's formula:

$$\left(1 - e^{-j2\pi n \frac{T_p}{T_s}}\right) = 1 - cos\left(-2\pi n \frac{T_p}{T_s}\right) - jsin\left(-2\pi n \frac{T_p}{T_s}\right)$$

Cosine is an odd function and sine is an even function:

$$\left(1 - e^{-j2\pi n \frac{T_p}{T_s}}\right) = 1 - cos\left(2\pi n \frac{T_p}{T_s}\right) + jsin\left(2\pi n \frac{T_p}{T_s}\right)$$

The object is to find the magnitude of the coefficients:

$$|c_n| = \left|\frac{-Aj}{2\pi n}\right| \cdot \left|1 - cos\left(2\pi n \frac{T_p}{T_s}\right) + jsin\left(2\pi n \frac{T_p}{T_s}\right)\right|$$

$$\left|\frac{-Aj}{2\pi n}\right| = \frac{A}{2\pi n}$$

To make our lives easier when dealing with this solution:

$$Let\ x = 2\pi n \frac{T_p}{T_s}$$

$$|1 - cosx + jsinx| = \sqrt{(1 - cosx)^2 + (sinx)^2} = \sqrt{1 - 2cosx + cos^2x + sin^2x}$$

Remembering trigonometry identities, and factoring out the 2:

$$|1 - cos(x) + jsin(x)| = \sqrt{2(1 - cosx)}$$

Using the Double Angle Formula:

$$cos2a = 1 - 2sin^2 a$$

Let $a = b/2$, and move the 1 to the cosine side:

$$1 - cosb = 2sin^2\frac{b}{2}$$

Plugging this back into our magnitude calculation and taking the square root gives:

$$\sqrt{2(1 - cosx)} = 2sin\frac{x}{2} = 2sin\left(\frac{2\pi n\frac{T_p}{T_s}}{2}\right) = 2sin\left(\pi n\frac{T_p}{T_s}\right)$$

Returning to our original equation:

$$c_n = \frac{-Aj}{2\pi n} \cdot \left(1 - e^{-j2\pi n\frac{T_p}{T_s}}\right)$$

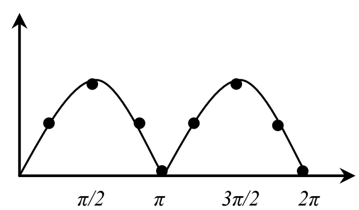
We have found the magnitude of c_n to be:

$$|c_n| = \frac{A}{2\pi n} \cdot \left|2sin\left(\pi n\frac{T_p}{T_s}\right)\right| = \frac{A}{\pi n} \cdot \left|sin\left(\pi n\frac{T_p}{T_s}\right)\right|$$

There was a lot of math to get to this point, but there was no magic. The reader should be able to get to this point, and we can now verify that if $T_p = T_s/2$ the non-zero even coefficients are zero.

$$\sin\left(\pi n\frac{T_s}{2T_s}\right) = sin\left(n\frac{\pi}{2}\right)$$

Refer to the magnitude plot of a sine wave below:



F-2 Magnitude plot of a sine wave, showing where to evaluate the sine wave for a 25% duty cycle

If we let T_p equal a quarter of a clock cycle we can determine the sine values of the coefficient by intuitively viewing F-2 or by using the formula:

$$c_n = \frac{A}{n\pi} \cdot \left|sin\left(\frac{n\pi}{4}\right)\right|$$

For a duty cycle of 25% the coefficient terms are:

$$c_n(25\%\ Duty\ Cycle) = \begin{cases} \dfrac{A}{4} = c_0 \\ \dfrac{A}{\pi} \cdot \sqrt{2} = c_1, c_5, c_9, c_{13}, \dots \\ \dfrac{A}{2\pi} \cdot 1 = c_2, c_6, c_{10}, c_{14}, \dots \\ \dfrac{A}{3\pi} \cdot \sqrt{2} = c_3, c_7, c_{11}, c_{15}, \dots \\ \dfrac{A}{4\pi} \cdot 0 = c_4, c_8, c_{12}, c_{16}, \dots \end{cases}$$

It should be intuitively easy to find what duty cycle is required to lose the 3rd harmonic by viewing the magnitude of the sine wave in F-2.

From this problem we can now answer the following questions:

- What is the spectrum display of a periodic function?
    - A train of pulses that are exponentially decaying
- What happens when the duty cycle is 50%?
    - We only see the odd pulses.
- What happens when the duty cycle is not 50%?
    - We will see the odd AND even pulses.

1.15 What is the Fourier transform of the signal seen in Fig. 1.29?

Solution:

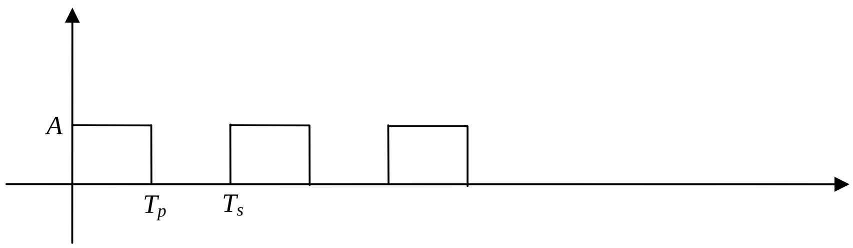


Fig. 1.29 Representing a squarewave using Exponential Fourier Series

The Fourier series for the periodic square wave is $g(t)=\sum_{n=-\infty}^{\infty} c_n e^{2\pi n f_0 t}$, and the coefficients, which are derived in the book, are

$$c_n = \frac{-A}{\pi n}\cdot j \quad \text{for} \quad n = 2k+1, \quad k = \cdots,-2,-1,0,1,2,\cdots$$

In these derivation, we used the fact that $\omega_0 T_s = 2\pi, T_s = 2T_p$.

The Fourier transform is given by

$$G(f)=\int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} c_n e^{2\pi nft}\right)e^{-j2\pi ft}dt = \sum_{n=-\infty}^{\infty}\left(\int_{-\infty}^{\infty} c_n e^{2\pi n f_0 t}\, e^{-j2\pi ft}dt\right) = \sum_{n=-\infty}^{\infty}\left(c_n\delta(f-nf_0)\right)$$

where $|c_n| = \frac{A}{\pi n}$, $n = \cdots,-3,-1,1,3,\cdots$, and $\angle c_n = \begin{cases} -90°, n>0 \\ 90°, n<0 \end{cases}$

Using the single-side spectrum with positive frequency only, we can rewrite the result as

$$G(f)=\sum_{n=1}^{\infty}\left(\frac{A}{\pi n}\delta(f-nf_0)\right)\cdot(-j), \quad n = 1,3,5,\cdots,2k+1,\cdots$$

where $|c_n| = \frac{A}{\pi n}$, and $\angle c_n = -90°$.

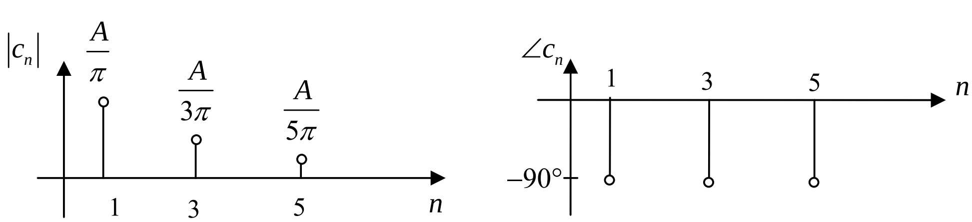


Figure Single-side spectrum of the square wave

Avinash Rajagiri

1.16) **What is the area under the Dirac delta function bordered by the x-axis? Why?**

Ans)

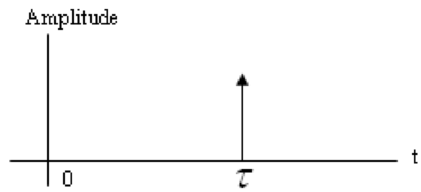


Figure: Dirac delta function

A Dirac delta function, is defined as $\delta(t-t_0)=\infty$. The function is nonzero at $t=t_0$ and zero for all other values of *t*. The area under the Dirac delta function bordered by the x-axis is equal to 1, the amplitude tends to go towards infinity as the width of the function goes towards zero.

This can be shown by using one of the important properties of the Dirac delta function, the sifting property.

$$x(\tau)=\int_{-\infty}^{\infty} x(t)\delta(t-\tau)dt \text{, for any value of } \tau \qquad (1)$$

So for the time-shifted version of the Dirac delta function above, $\delta(t-\tau)$ is nonzero only at $t=\tau$ and zero for all other values of t. Rewriting the above equation for a non-time shifted delta function, $\delta(t)$ that exists at $t=\tau=0$ we get,

$$x(0)=\int_{-\infty}^{\infty} x(t)\delta(t)dt \qquad (2)$$

For a special case, when the function $x(t)$ is a constant say, 1 for all values of t. We can write eq(1) and eq(2) as,

$$1=\int_{-\infty}^{\infty} \delta(t)dt=\int_{-\infty}^{\infty} \delta(t-\tau)dt \qquad (3)$$

As we know the integral of a function defines the area underneath it, we can conclude that the area underneath the Dirac delta function must be equal to unity.

Jason Durand

Problem 1.17 – Show how to take the Fourier transform of $\sin(2\pi f_0 t+\theta)$ and $\cos(2\pi f_0 t+\theta)$. Plot the magnitude and phase responses of the transforms.

The Fourier transform of a signal is given by $G(f)=\int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}dt$, and this equation is easier to work with if we first convert $\sin(2\pi f_0 t+\theta)$ into exponential form with Euler's identity.

From Euler's identity: $e^{-j\theta}=\cos\theta+j\sin\theta$,

$$\sin(2\pi f_0 t+\theta) = \frac{e^{j2\pi f_0 t+\theta}-e^{-(j2\pi f_0 t+\theta)}}{2j} = \frac{e^{j\theta}}{2j}e^{j2\pi f_0 t}-\frac{e^{-j\theta}}{2j}e^{-j2\pi f_0 t} .$$

Next, use the Fourier transform integral.

$$G(f) = \frac{e^{j\theta}}{2j}\int_{-\infty}^{\infty} e^{j2\pi f_0 t}e^{-j2\pi f t}dt - \frac{e^{-j\theta}}{2j}\int_{-\infty}^{\infty} e^{-j2\pi f_0 t}e^{-j2\pi f t}dt$$

$$G(f) = \frac{e^{j\theta}}{2j}\int_{-\infty}^{\infty} e^{-j2\pi t(f-f_0)}dt - \frac{e^{-j\theta}}{2j}\int_{-\infty}^{\infty} e^{-j2\pi t(f+f_0)}dt \quad \text{(pay close attention to signs)}$$

Recall that the Fourier transform of a constant is the delta function in the frequency domain, or that

$$Fourier(C) = \int_{-\infty}^{\infty} C e^{-j2\pi f t}dt = C\delta(f) .$$

The exponentials in the denominators are simply the polar forms of the imaginary number, to make it easier to divide (rectangular form for adding/subtracting, polar form for multiplying/dividing).

$$G(f) = \frac{e^{j\theta}}{2e^{j\frac{\pi}{2}}}\delta(f-f_0)-\frac{e^{-j\theta}}{2e^{j\frac{\pi}{2}}}\delta(f+f_0)$$

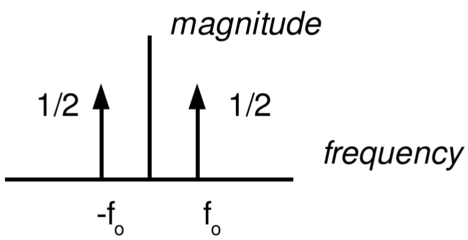
$$G(f) = \frac{1}{2}e^{j(\theta-\frac{\pi}{2})}\delta(f-f_0)-\frac{1}{2}e^{-j(\theta+\frac{\pi}{2})}\delta(f+f_0)$$

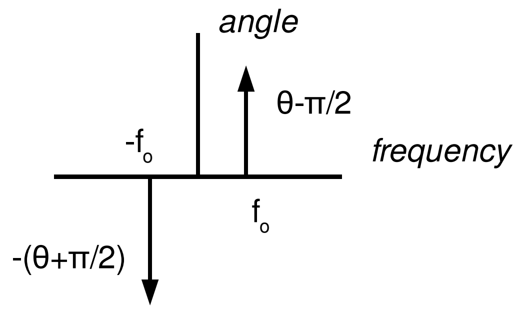
Magnitude and phase

$$|G(f)| = \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$$

$$\measuredangle G(f) = (\theta-\frac{\pi}{2})\delta(f-f_0) - (\theta+\frac{\pi}{2})\delta(f+f_0)$$

Magnitude and Phase plots:





Next, for the Fourier transform of $\cos(2\pi f_0 t+\theta)$, most of the same steps are used, and it is redundant to repeat all of them. Using Euler's Identity, $\cos(2\pi f_0 t+\theta)$ is rewritten as

$$\cos(2\pi f_0 t+\theta) = \frac{e^{j2\pi f_0 t+\theta}+e^{-(j2\pi f_0 t+\theta)}}{2} = \frac{e^{j\theta}}{2}e^{j2\pi f_0 t}+\frac{e^{-j\theta}}{2}e^{-j2\pi f_0 t}$$

Fourier transform integral:

$$G(f)=\int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}dt$$
$$G(f) = \frac{e^{j\theta}}{2}\int_{-\infty}^{\infty} e^{j2\pi f_0 t}e^{-j2\pi f t}dt + \frac{e^{-j\theta}}{2}\int_{-\infty}^{\infty} e^{-j2\pi f_0 t}e^{-j2\pi f t}dt$$
$$G(f) = \frac{e^{j\theta}}{2}\int_{-\infty}^{\infty} e^{-j2\pi t(f-f_0)}dt + \frac{e^{-j\theta}}{2}\int_{-\infty}^{\infty} e^{-j2\pi t(f+f_0)}dt$$
$$G(f) = \frac{1}{2}e^{j\theta}\delta(f-f_0)+\frac{1}{2}e^{-j\theta}\delta(f+f_0)$$

Magnitude and Phase

$$|G(f)| = \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$$

$$\measuredangle G(f) = \theta\,\delta(f-f_0) - \theta\,\delta(f+f_0)$$

Magnitude and Phase plots