Chapter

1

Signals, Filters, and Tools

Mixed-signal circuit design requires a fundamental knowledge of signals, signal processing, and circuit design. In this chapter we provide an overview of signals, filtering, and the mathematical tools. The chapter may be a review for the reader; however, we use it to ensure a good foundation to build on in the coming chapters and to provide a quick reference for the mathematical formulas we'll use throughout the book.

1.1 Sinusoidal Signals

Let's take a fundamental look at the sinewave. While there are many ways (equations and formulas) of representing a sinewave, we must remember it is an empirically determined function. Naturally occurring signals, shapes, or constants are determined or described through empirical measurements or observations. For example, π is determined by dividing the circumference of a circle by its diameter

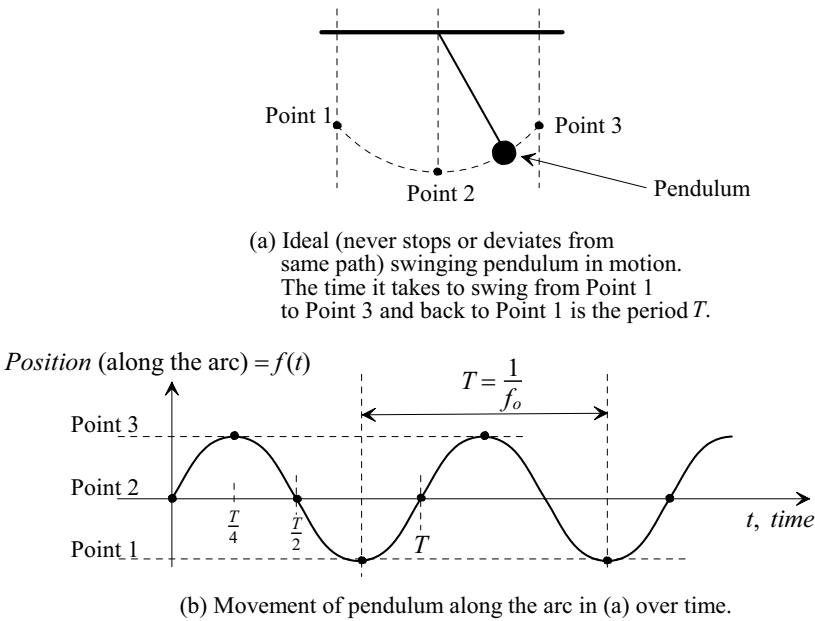$$\pi = \frac{\text{circumference}}{\text{diameter}} \qquad (1.1)$$

The goal of this section is to provide intuitive discussions that will help create a deeper understanding of what's going on in a circuit or system.

1.1.1 The Pendulum Analogy

Consider the (ideal, that is, lossless) moving pendulum seen in Fig. 1.1a. In this figure the pendulum is moving back and forth between Points 1 and 3 repeatedly over time. As the pendulum leaves Point 1 it starts out slow, gaining maximum speed as it passes Point 2, and finally reaching Point 3. At Point 3 it stops and reverses direction. The time it takes to make this complete journey back to the starting point, Point 1 in this discussion, is the period, T. In Fig. 1.1b we plot the movement of the pendulum along the arched path. We record the position to define a function, $f(t)$, that indicates the pendulum's position at a specific time

$$Position = f(t) = f(t + nT), \text{ where } n \text{ is an integer} \qquad (1.2)$$

This signal, we should all recognize, is a sinusoid or sinewave which repeats its position with a frequency, f_o, of $1/T$.

(a) Ideal (never stops or deviates from
    same path) swinging pendulum in motion.
    The time it takes to swing from Point 1
    to Point 3 and back to Point 1 is the period T.

(b) Movement of pendulum along the arc in (a) over time.

Figure 1.1 Physical interpretation of a sinewave.

Next, consider the circle seen in Fig. 1.2. One complete rotation around this circle (360 degrees or 2π) is analogous to one complete movement (swing) of our pendulum. We started plotting the pendulum's position at Point 2 in Fig. 1.1b (Point 2, $t = 0$, in Fig. 1.2). After $T/4$ we reach Point 3 in Fig. 1.1b. This corresponds to a 90 degree, or $\pi/2$, movement in our circle. After another $T/4$ seconds we pass back through Point 2. In the circle we've moved 180 degrees. This continues with each swing of the pendulum corresponding to a complete revolution around the circle. Note that we do have some
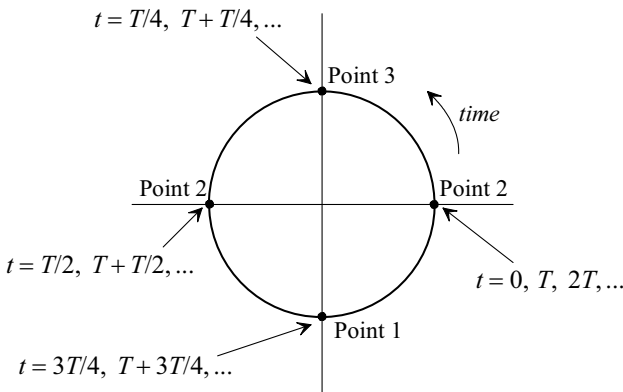
**Figure 1.2** Using a circle to describe the movement of the pendulum in Fig. 1.1.

limitations when representing the movement of the pendulum with this circle. For example, what is the amplitude of the sinewave (what is the relative position of the pendulum along the arched path)? We'll address these concerns in a moment. For now let's write, assuming we are using radian angular units,

$$Position = \sin\left(2\pi \cdot \frac{t}{T}\right) = \sin(2\pi f_o \cdot t) \tag{1.3}$$

This function, the sine function, tells us our relative position along the arc (the argument of this function is the angle which relates to the position on the circle in Fig. 1.2). Point 2 corresponds to the function having a value of 0 (and times, $t = 0$, $T/2$, T, $3T/2$, $2T$, ...), Point 3 to a value of +1, and Point 1 corresponds to −1. Finally, remember that the values of the sine function in Fig. 1.1.b, and Eq. (1.3), are determined empirically from measured data (e.g., plotting the pendulum's position along the arched path against time).

Describing Amplitude in the x-y Plane

Examine the sinewave in Fig. 1.3a. For the moment we won't concern ourselves with the actual distance the pendulum swings. In Fig. 1.3b we represent the sinewave, at Point i (and Point vi), as a zero length vector along the *x*-axis (the amplitude of the sinewave is 0 at this point in time). As we move towards Point ii in Fig. 1.3a the length of the vector
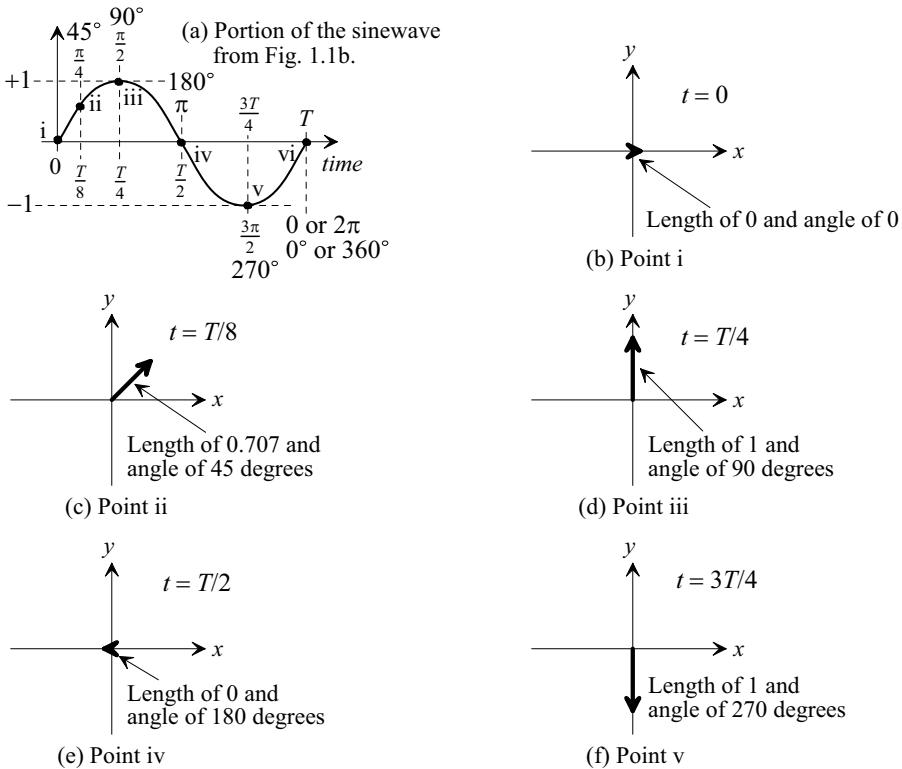
Figure 1.3 A vector swinging around the x-y plane changing both length and angle is used to represent a sinewave.

increases (here indicating an increase in both the x and y directions), Fig. 1.3c. At $T/4$ we are at Point iii in Fig. 1.3a. As seen in Fig. 1.3d the length of the vector is 1 and the angle is 90 degrees. Continuing on towards $T/2$ (Point iv) in Fig. 1.3a the vector is shrinking once again finally reaching a length of 0 and an angle of 180 degrees, Fig. 1.3e. In Fig. 1.3f, Point v, the length is 1 and the angle is 270 degrees. The **key point** here is that a sinewave is represented in the x-y plane by a vector that is changing length and *rotating* around in a circle. Knowing this we can simply represent the sinewave by its peak value and the associated angle between the x-axis and the vector as indicated in Fig. 1.3. For example, as seen in Fig. 1.3d, the peak value of the sinewave occurs when the angle is 90 degrees ($T/4$) so we could write

$$\sin 2\pi f_o \cdot t \rightarrow 1 \angle 90° \rightarrow x = 0, y = 1 \qquad (1.4)$$

knowing the vector representing the sinewave is actually rotating around in the x-y plane with time (and a frequency f_o). Note that we could plot the sinewave more accurately by adding a third axis, time (the z-axis), and showing the corresponding, corkscrew looking, 3-dimensional plot.

In-Phase and Quadrature Signals

Consider moving the sinusoid in Fig. 1.3a a quarter of a cycle (90 degrees, $\pi/2$, or a delay, t_{delay}, of $T/4$) earlier in time, Fig. 1.4a. We write, for this time-shifted signal,

$$\sin\left(\frac{2\pi}{T} \cdot t + \frac{\pi}{2}\right) = \sin\left(\frac{360}{T} \cdot t + 90\right) = \sin\left(\frac{2\pi}{T} \cdot \left(t + \frac{T}{4}\right)\right) = \sin\left(2\pi f_o \cdot t + \frac{\pi}{2}\right) \qquad (1.5)$$

As the reader probably already knows, this signal function is called cosinusoidal or simply a cosine signal and is described using

$$\sin\left(2\pi f_o \cdot t + \frac{\pi}{2}\right) = \cos\left(2\pi f_o \cdot t\right) \qquad (1.6)$$

We can say that the cosine signal in Fig. 1.4a *leads* the sine signal in Fig. 1.3a by a phase shift, θ, of 90 degrees (= $T/4$ = $\pi/2$). We could also say that the sine signal *lags* the cosine signal by a phase shift of −90 degrees. Note that when we talk about phase shift it's assumed that the frequencies of the two signals are equal. It doesn't make sense to talk about the phase shift between two sinusoids at different frequencies. Finally note that the phase shift is given by

$$\theta = 2\pi \cdot \frac{t_{delay}}{T} = 2\pi f_o \cdot t_{delay} \qquad (1.7)$$

where $t_{delay}/T \times 100\%$ is the percentage the delay is of the period.

Figure 1.4b shows the x-y plane plot for a cosine signal. The peak value of the cosine signal occurs when the angle is 0 degrees so we could write

$$\cos 2\pi f_o \cdot t \rightarrow 1 \angle 0° \rightarrow x = 1, y = 0 \qquad (1.8)$$

to represent the signal and not show the rotating vector with time. Note that we can say the cosine signal is *in-phase* (*I*) since we are representing it above with a zero degree phase shift. We could also say that the sine signal, since it's shifted in time by a quarter cycle, Eq. (1.4), has a *quadrature* (*Q*) phase shift. Note what happens, looking at the vector representations of the sinusoids in Figs. 1.3b-f and 1.4b, if we create a signal by *adding a sine signal to a cosine signal* or
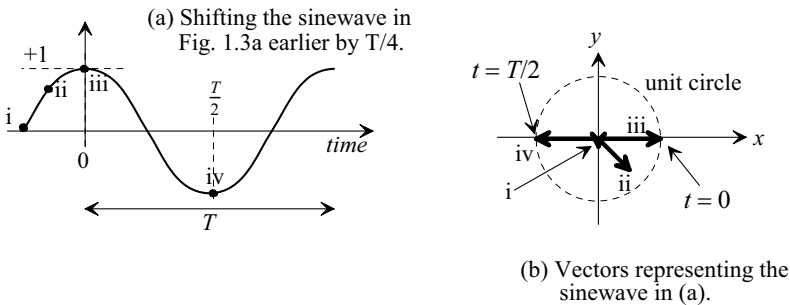
$$S_{IQ}(t) = \cos 2\pi f_o \cdot t + \sin 2\pi f_o \cdot t \qquad (1.9)$$

The resulting signal, when plotted in the x-y plane (Fig. 1.5), results in a vector (sinewave) with a length (peak amplitude) of $\sqrt{1^2 + 1^2}$ or $\sqrt{2}$ that simply rotates around in a circle with a frequency of $1/T\,(=f_o)$. In more general terms, Fig. 1.6,

$$S_{IQ}(t) = A_I \cdot \cos 2\pi f_o \cdot t + A_Q \cdot \sin 2\pi f_o \cdot t = \sqrt{A_I^2 + A_Q^2} \cdot \cos\left(2\pi f_o \cdot t + \tan^{-1}\frac{A_Q}{A_I}\right) \qquad (1.10)$$

where we define the magnitude and phase as

$$|S_{IQ}(t)| \angle S_{IQ}(t) = \sqrt{A_I^2 + A_Q^2} \angle \tan^{-1}\frac{A_Q}{A_I} \qquad (1.11)$$

(a) Shifting the sinewave in Fig. 1.3a earlier by T/4.
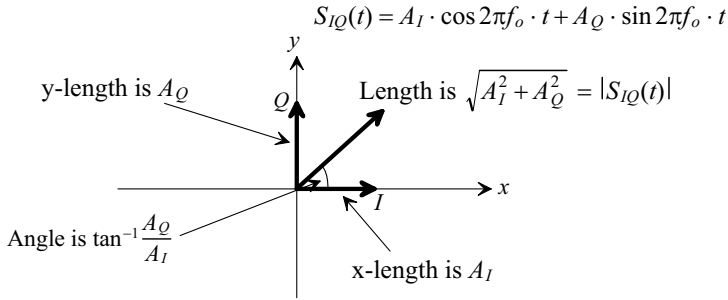
(b) Vectors representing the sinewave in (a).

Figure 1.4 Shifting the sinewave in Fig. 1.3a earlier in time by T/4.

Before going too much further, we should ask why we would want to add I and Q signals? The answer is that since the I/Q sinusoids are at the same frequency, shifted in time by a quarter cycle, we can transmit them with, for example, changes in their respective amplitudes and increase the information sent for a given bandwidth. Note that we can't shift the sinewave seen in Fig. 1.3a by 180 degrees, or $T/2$, and add it to an unshifted sinewave since we would get no signal at all!
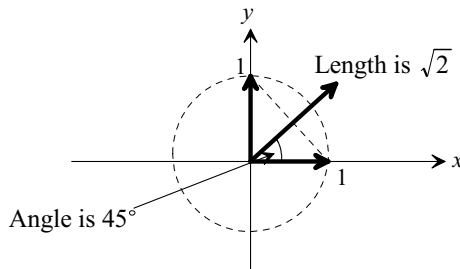
Figure 1.5 Showing how an I/Q signal can be represented in the x-y plane.

Notice that the signals here are very simple. What happens when we start multiplying them together and shifting them in time (changing the phase shift)? We need

$$S_{IQ}(t) = A_I \cdot \cos 2\pi f_o \cdot t + A_Q \cdot \sin 2\pi f_o \cdot t$$

y-length is A_Q

Length is $\sqrt{A_I^2 + A_Q^2} = |S_{IQ}(t)|$

Angle is $\tan^{-1} \dfrac{A_Q}{A_I}$

x-length is A_I

Figure 1.6 Again, showing how an I/Q signal can be represented in the x-y plane.

to simplify the math! To move towards this goal we'll develop the complex, or *z*-, plane and the frequency-domain representation of signals.

1.1.2 The Complex (*z*-) Plane

Let's attempt (and fail using the *x-y* plane) to simplify our mathematical description of the *IQ* signal given in Eq. (1.10). Recall the following Taylor series expansions

$$e^k = 1 + k + \frac{k^2}{2!} + \frac{k^3}{3!} + \frac{k^4}{4!} + \dots \tag{1.12}$$

$$\cos k = 1 - \frac{k^2}{2!} + \frac{k^4}{4!} - \frac{k^6}{6!} + \frac{k^8}{8!} - \dots \tag{1.13}$$

$$\sin k = k - \frac{k^3}{3!} + \frac{k^5}{5!} - \frac{k^7}{7!} + \frac{k^9}{9!} - \dots \tag{1.14}$$

We can now write

$$\cos k + \sin k = 1 + k - \frac{k^2}{2!} - \frac{k^3}{3!} + \frac{k^4}{4!} + \frac{k^5}{5!} - \frac{k^6}{6!} - \frac{k^7}{7!} + \frac{k^8}{8!} + \frac{k^9}{9!} - \dots \tag{1.15}$$

Comparing Eq. (1.15) to Eq. (1.12) we see that we are close to writing the Taylor's series for e^k. Why is this important? Perhaps the simplest explanation is that if we can represent sinewaves using exponentiation, then multiplying two sinewaves, or shifting a sinewave in time, can be performed using simple addition (of exponents).

The question now is how do we modify things to ensure that all terms are added so that Eq. (1.15) matches Eq. (1.12)? Let's look at the first discrepancy $(-1) \cdot \frac{k^2}{2!}$. The only way to change the polarity of this term is take the square root of -1 and move it inside with k^2. As the reader may know instead of writing $\sqrt{-1}$ for all of these terms we simplify things and write

$$j = \sqrt{-1} \tag{1.16}$$

Numbers using j (or i) are called *imaginary or complex numbers* (the reason for using the name imaginary will be explained in Ex. 1.1). Imaginary numbers are invaluable for time-shifting and scaling sinusoidal signals. We now rewrite Eq. (1.12) using j as

$$e^{jk} = 1 + jk - \frac{k^2}{2!} - j\frac{k^3}{3!} + \frac{k^4}{4!} + j\frac{k^5}{5!} - \frac{k^6}{6!} - j\frac{k^7}{7!} + \dots \tag{1.17}$$

or Euler's formula

$$e^{jk} = \cos k + j \cdot \sin k \tag{1.18}$$

where the cosine term is a real number and the sine term is the imaginary component of the complex number. Sometimes the following notation is used

$$Re\{e^{jk}\} = \cos k \text{ and } Im\{e^{jk}\} = \sin k \tag{1.19}$$

Further, with a little algebraic manipulation we can also write

$$\cos k = \frac{e^{jk} + e^{-jk}}{2}, \quad \sin k = \frac{e^{jk} - e^{-jk}}{2j} \tag{1.20}$$

The next thing we have to discuss is plotting complex numbers. Examine Fig. 1.7. Plotting the real component, x, of a complex number, $x + jy$, follows the same methods we've always used. To plot the imaginary component, we now use the y-axis. Note that the factor j in the complex number simply indicates the imaginary component and shouldn't be included when finding the magnitude of the number, $\sqrt{x^2 + y^2}$ or the phase, θ, of the complex number, $\tan^{-1}\frac{y}{x}$.
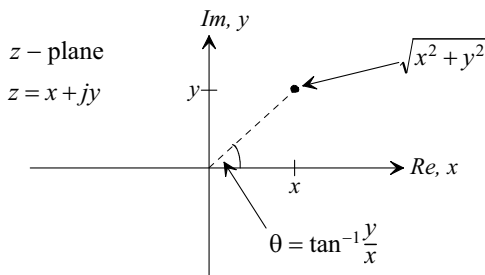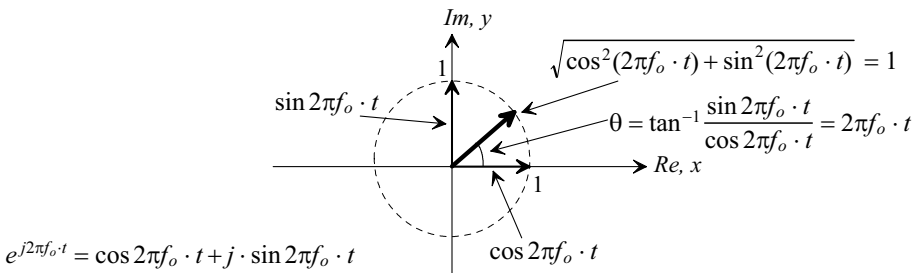

Figure 1.7 The complex plane, plotting the imaginary number

Example 1.1
In the complex plane, plot the signal $e^{j2\pi f_o \cdot t}$. Comment on the resulting plot.

As seen in Fig. 1.8 the magnitude is $|e^{j2\pi f_o \cdot t}| = 1$ and phase shift is $\angle e^{j2\pi f_o \cdot t} = 2\pi f_o \cdot t$. The signal simply swings around and around in the complex

Figure 1.8 Plotting Euler's formula in the z-plane.

plane, on the unit circle, without changing amplitude (both the real and the imaginary components oscillate back and forth between +1 and −1). One complete rotation takes $1/f_o$, or T seconds. We need to pause a moment and ask "If the magnitude of $e^{j2\pi f_o \cdot t}$ is a constant value of 1 isn't it a DC signal?" (answer: no) When a DC signal is represented in the x-y plane it doesn't rotate or change amplitude like a sinewave does. (Well, since the frequency of a DC signal, f, is 0, you could say it does rotate around the x-y plane just like a sinewave but since $T = \infty$ $(=1/f)$ it never leaves the x-axis). What this, $e^{j2\pi f_o \cdot t}$, **imaginary signal** can be used for is to introduce or represent delay (phase shift). ∎

To understand this last statement in more detail let's write

$$A \cdot \cos 2\pi f_o \cdot t = Re\{A \cdot e^{j2\pi f_o \cdot t}\} \rightarrow A \angle 0 \qquad (1.21)$$

Suppose we want to delay this signal by t_d. We can represent a delay at a particular frequency, f_o, using

$$e^{j2\pi f_o \cdot (-t_d)} \rightarrow 1 \angle 2\pi f_o \cdot (-t_d) \rightarrow 1 \angle 2\pi \cdot \frac{-t_d}{T} \qquad (1.22)$$

The delayed cosine signal can be written as

$$A \cdot \cos 2\pi f \cdot (t - t_d) = Re\{A \cdot e^{j2\pi f \cdot t} \cdot e^{j2\pi f \cdot (-t_d)}\} = Re\{A \cdot e^{j2\pi f \cdot (t-t_d)}\} \qquad (1.23)$$

To *simplify the notation* we can drop the real indication, $Re\{\}$. We could also describe the delayed cosine signal in terms of angle notation as

$$A \angle 0 \cdot 1 \angle 2\pi \cdot \frac{-t_d}{T} = A \cdot 1 \angle \left[0 + 2\pi \cdot \frac{-t_d}{T} \right] = A \angle (-2\pi f_o \cdot t_d) \qquad (1.24)$$
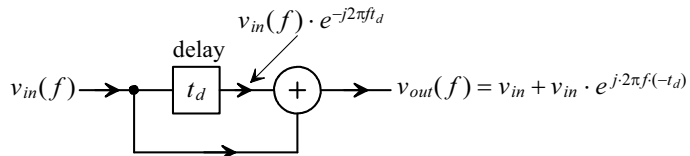
Again noting that when we use angle notation, the x-y plane, or the complex plane, we need to know the frequency, f_o, of the input signal since this information isn't present in these representations.

1.2 Comb Filters

A delay can be used to construct a simple, but very useful, filter called a comb filter, Fig. 1.9. Before deriving the equations that characterize this filter let's discuss notation. If our input signal is a sinusoid we can represent it using

$$v_{in}(t) = A \cos 2\pi f \cdot t \qquad (1.25)$$

If we were to keep the frequency fixed, $f = f_o$, and vary time, which we've done up to this point, the output signal of a circuit will simply repeat itself (at the same frequency as the input signal). *What's more useful* is varying the input signal's frequency, f, and looking at the output of the circuit or system (how the amplitude and phase shift vary). To move

Figure 1.9 A comb filter.